

# Contact Mechanics and Elements of Tribology

## Lecture 1. *Mechanical Contact*

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@ Centre des Matériaux (& virtually)  
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## Lecture 1

- 1 Balance equations
- 2 Intuitive notions
- 3 Formalization of frictionless contact
- 4 Flamant's solution
- 5 Integration of Flamant' solution
- 6 Displacements and tractions
- 7 Contact types
- 8 Analogy with boundary conditions

## Lecture 2

- 1 Evidence of friction
- 2 Friction models
- 3 Boussinesq, Cerruti
- 4 Hertzian contact
- 5 Classical contact problems

# Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_{-v} = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \underline{u} \nabla)$$

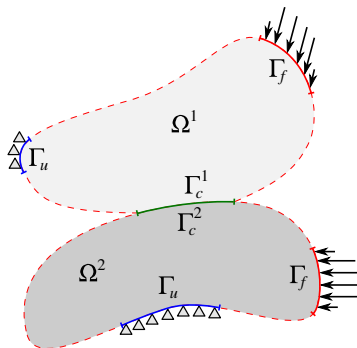
- Constitutive equation

$$\underline{\underline{\sigma}} = W'(\underline{\underline{\varepsilon}})$$

- Boundary conditions

$$\text{Dirichlet: } \underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$$

$$\text{Neumann: } \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$$



Two bodies in contact

# Boundary value problem in elasticity

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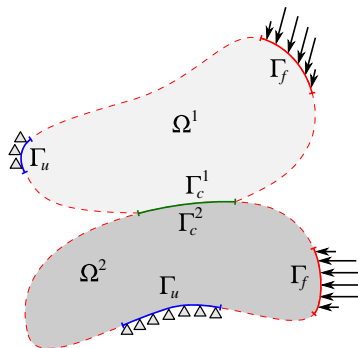
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Two bodies in contact

- **Include contact conditions**

...

# Intuitive conditions

- 1 No penetration

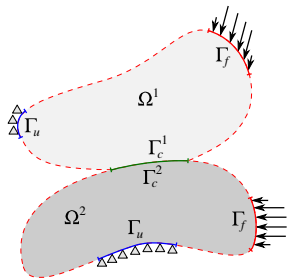
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

- 2 No adhesion

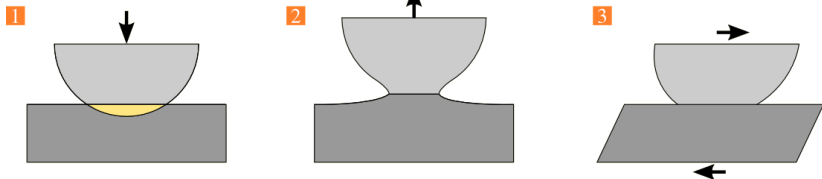
$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \leq 0, \forall \underline{x} \in \Gamma_c^i$$

- 3 No shear stress

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$



Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

# Intuitive conditions

- 1 No penetration

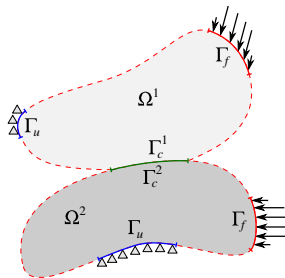
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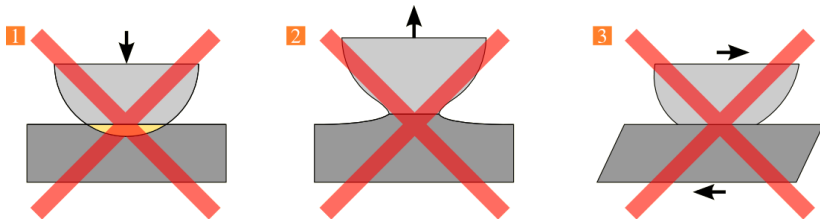
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Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

# Contact problem

## ≈ Problem

Find such contact pressure

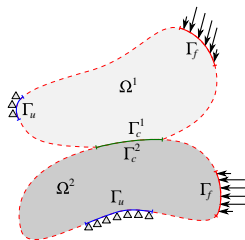
$$p = -\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \geq 0$$

which being applied at  $\Gamma_c^1$  and  $\Gamma_c^2$  results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$



Two bodies in contact

- Unfortunately, we do not know  $\Gamma_c^1$  in advance, it is also an unknown of the problem.

## ■ Related problem

Suppose that we know  $p$  on  $\Gamma_c$

Then what is the corresponding displacement field  $\underline{u}$  in  $\Omega^i$ ?

Or the other way around?

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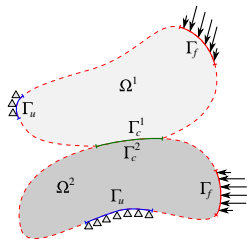
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See Flamant problem TD



- Normal force (in-plane stresses and displacements (plane strain))

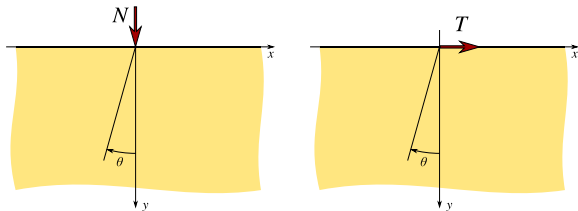
$$\sigma_r = -\frac{2N}{\pi} \frac{\cos(\theta)}{r} \quad \text{or} \quad \sigma_x = -\frac{2N}{\pi} \frac{x^2 y}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2N}{\pi} \frac{y^3}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2N}{\pi} \frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N \cos(\theta) [2(1-\nu) \ln(r) - (1-2\nu)\theta \tan(\theta)] + C \cos(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} N \sin(\theta) [2(1-\nu) \ln(r) - 2\nu + (1-2\nu)(1-2\theta \cotan(\theta))] - C \sin(\theta)$$

- On the surface

$$u_x = -\frac{N(1+\nu)(1-2\nu)}{2E} \text{sign}(x), \quad u_y = \frac{2N(1-\nu^2)}{\pi E} \log(|x|) + C$$



- Tangential force (in-plane stresses and displacements (plane strain))

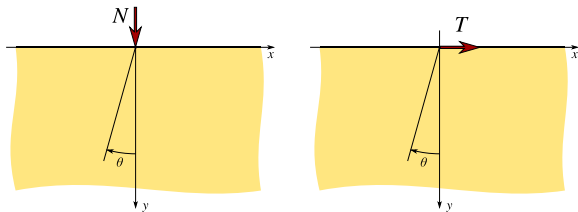
$$\sigma_r = \frac{2T \sin(\theta)}{\pi r} \quad \text{or} \quad \sigma_x = -\frac{2T}{\pi} \frac{x^3}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2T}{\pi} \frac{xy^2}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2T}{\pi} \frac{x^2y}{(x^2+y^2)^2}$$

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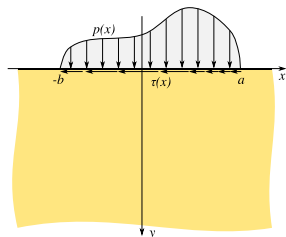
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$$u_x = -\frac{2T(1-\nu^2)}{\pi E} \log(|x|) + C, \quad u_y = \frac{T(1+\nu)(1-2\nu)}{2E} \text{sign}(x)$$



# Distributed load

- Distributed tractions  $p(x)dx = dN(x)$ ,  
 $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



*Tractions on the surface*

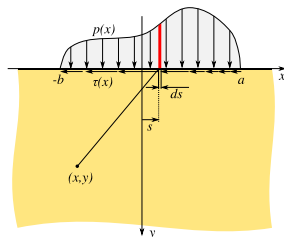
$$\sigma_x(x, y) = -\frac{2y}{\pi} \int_{-b}^a \frac{p(s)(x-s)^2 ds}{((x-s)^2 + y^2)^2} - \frac{2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s)^3 ds}{((x-s)^2 + y^2)^2}$$

$$\sigma_y(x, y) = -\frac{2y^3}{\pi} \int_{-b}^a \frac{p(s) ds}{((x-s)^2 + y^2)^2} - \frac{2y^2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s) ds}{((x-s)^2 + y^2)^2}$$

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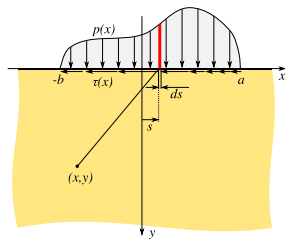
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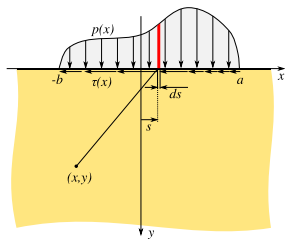


*Tractions on the surface*

$$u_x(x, 0) = -\text{sign}(x) \frac{(1-2\nu)(1+\nu)}{2E} \left[ \int_{-b}^x p(s) ds - \int_x^a p(s) ds \right] - \frac{2(1-\nu^2)}{\pi E} \int_{-b}^a \tau(s) \ln|x-s| ds + C_1$$

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- Or rather their derivatives along the surface



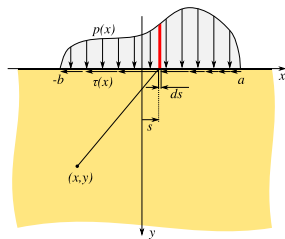
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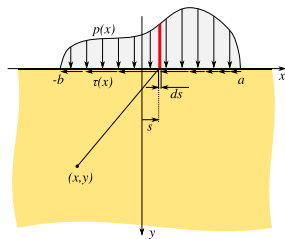


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- Link displacement derivatives with tractions

$$\int_{-b}^a \frac{\tau(s)}{x-s} ds = -\frac{\pi(1-2\nu)}{2(1-\nu)} p(x) - \frac{\pi E}{2(1-\nu^2)} u_{x,x}(x, 0)$$

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- If in contact interface we can prescribe  $p, u_{x,x}$  or  $\tau, u_{y,x}$ , then the problem reduces to

$$\int_{-b}^a \frac{\mathcal{F}(s)}{x-s} ds = \mathcal{U}(x)$$

- The general solution (case  $a = b$ ):

$$\mathcal{F}(x) = \frac{1}{\pi^2 \sqrt{a^2 - x^2}} \int_{-a}^a \frac{\sqrt{a^2 - s^2} \mathcal{U}(s) ds}{x-s} + \frac{C}{\pi \sqrt{a^2 - x^2}}, \quad C = \int_{-a}^a \mathcal{F}(s) ds$$

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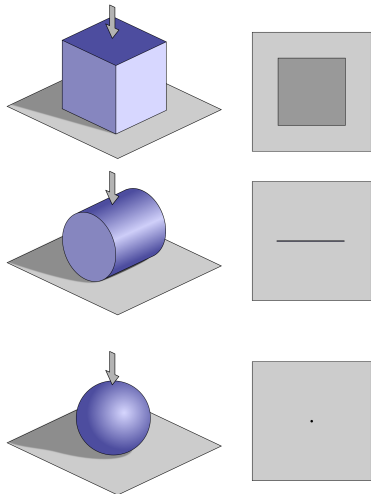
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flat frictionless punch, consider P.V.

# Types of contact

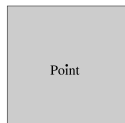
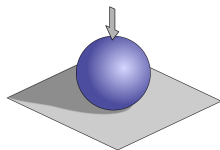
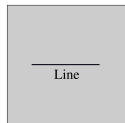
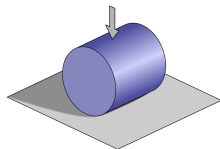
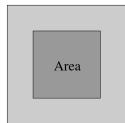
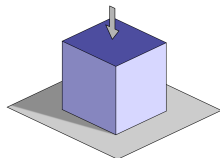
- Known contact zone
  - conformal geometry  
*flat-to-flat, cylinder in a hole*
  - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone  
*general case*
- Point and line contact
- Frictionless  
*conservative, energy minimization problem*
- Frictional  
*path-dependent solution, from the first touch to the current moment*



Example

# Types of contact

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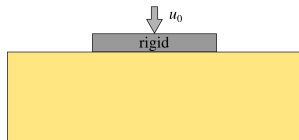
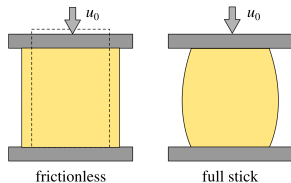
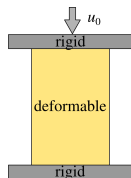


Example

# Analogy with boundary conditions

## Flat geometry

- Compression of a cylinder
- Frictionless  $u_z = u_0$
- Full stick conditions  $\underline{u} = u_0 \underline{e}_z$
- Rigid flat indenter  $u_z = u_0$



# Analogy with boundary conditions

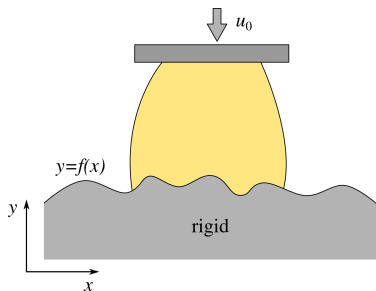
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## Curved geometry

- Polar/spherical coordinates  
 $u_r = u_0$
- If frictionless contact on rigid surface  $y = f(x)$  is retained by high pressure

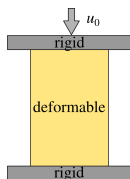
$$(\underline{X} + \underline{u}) \cdot \underline{e}_y = f((\underline{X} + \underline{u}) \cdot \underline{e}_x)$$



# Analogy with boundary conditions

## Flat geometry

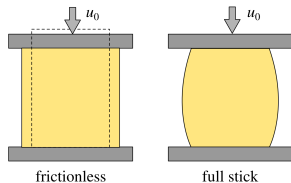
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## Curved geometry

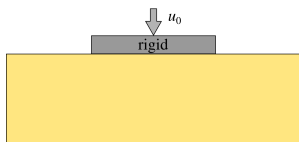
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## Transition to finite friction

-   $\approx$  From full stick, decrease  $f$  by keeping  $u_z = 0$  and by replacing in-plane Dirichlet BC by in-plane Neumann BC



# Analogy with boundary conditions II

## In general

- Type I: prescribed tractions

$$p(x, y), \tau_x(x, y), \tau_y(x, y)$$

- Type II: prescribed displacements

$$\underline{u}(x, y)$$

- Type III: tractions and displacements

$$u_z(x, y), \tau_x(x, y), \tau_y(x, y) \text{ or}$$

$$p(x, y), u_x(x, y), u_y(x, y)$$

- Type IV: displacements and relation between tractions

$$u_z(x, y), \tau_x(x, y) = \pm f p(x, y)$$







To be continued. . .

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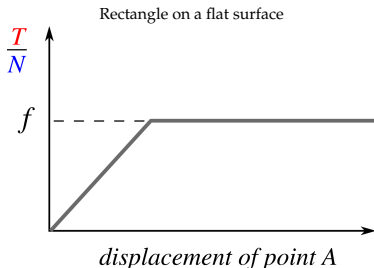
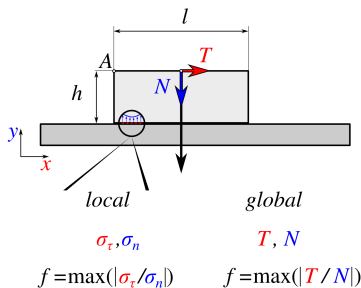
# Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area



*Think about adhesion and introduce a threshold in the interface  $\tau_c$*

- Globally:
  - stick:  $T < T_c(N)$
  - slip:  $T = T_c(N)$
- From experiments:
  - Threshold  $T_c \sim N$
  - Friction coefficient  $f = |T_c/N|$
- Locally
  - stick:  $\sigma_\tau < \tau_c(\sigma_n)$
  - slip:  $\sigma_\tau = f\sigma_n$



# Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area

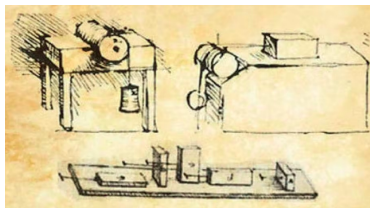


*Think about adhesion and introduce a threshold in the interface  $\tau_c$*

- Globally:
  - stick:  $T < T_c(N)$
  - slip:  $T = T_c(N)$
- From experiments:
  - Threshold  $T_c \sim N$
  - Friction coefficient  $f = |T_c/N|$
- Locally
  - stick:  $\sigma_\tau < \tau_c(\sigma_n)$
  - slip:  $\sigma_\tau = f\sigma_n$



Torque



First ever frictional experiments:  
notebook drawings of Leonardo  
da Vinci

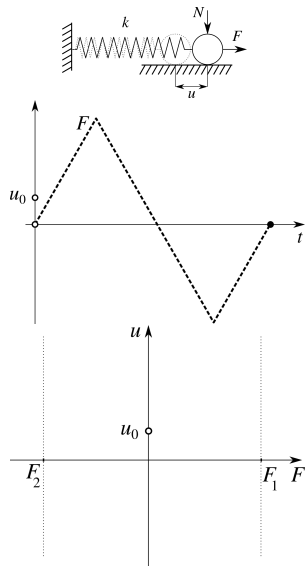
- proportionality between weight and frictional force
- friction is independent on the contact area

# Friction is not that intuitive

- Frictionless  
*conservative, energy minimization problem*
- Frictional  
*path-dependent solution, from the first touch to the current moment*



Example



# Friction is not that intuitive

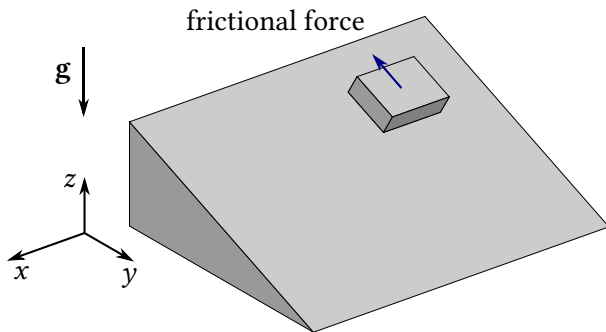
What does your intuition says?



When the force starts to decrease, the contact point  
 goes to the left    does not move    goes to the right

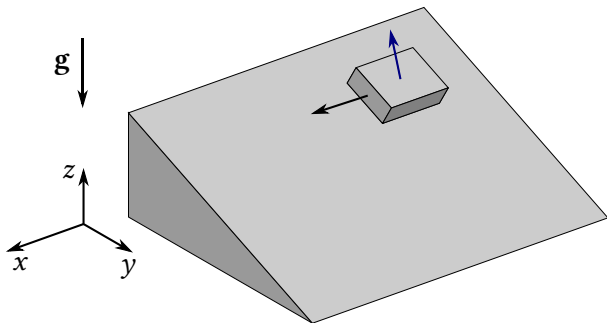
# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



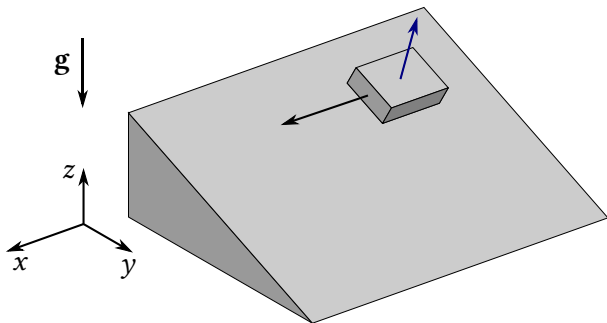
# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



# Direction of sliding

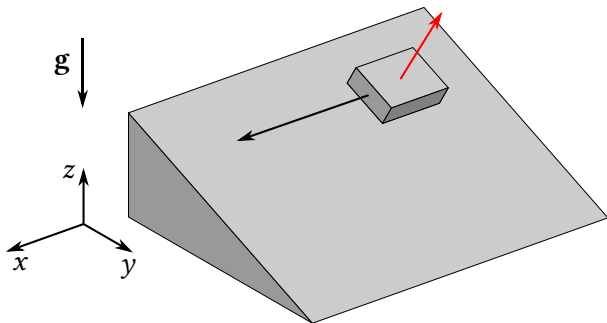
Sliding orthogonally to the slope of the inclined plane





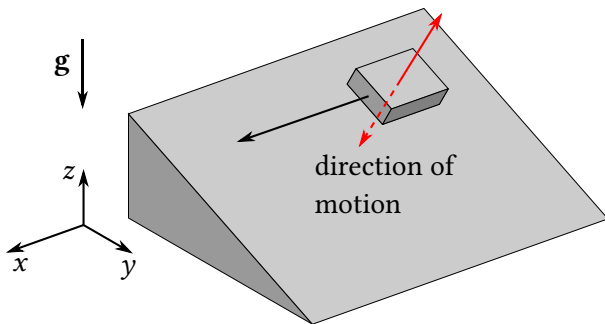
# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



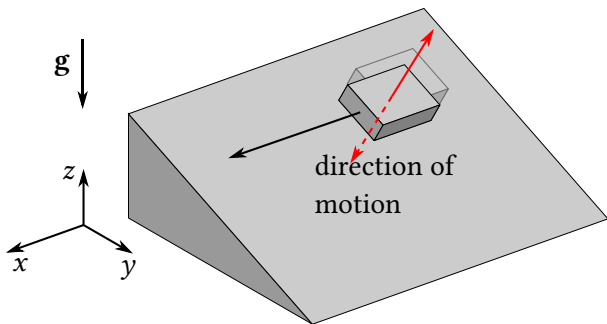
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Sliding orthogonally to the slope of the inclined plane

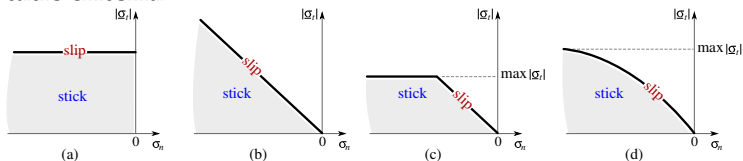


# Direction of sliding

Sliding orthogonally to the slope of the inclined plane

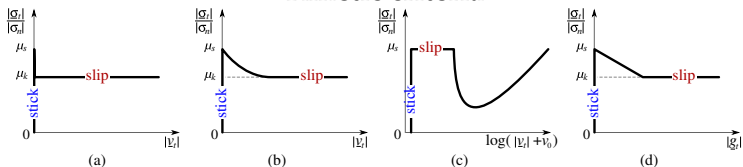


## • Static criteria



(a) Tresca    (b) Amontons-Coulomb    (c) Coulomb-Orowan    (d) Shaw

## • Kinetic criteria



(a,b) velocity weakening    (c) velocity weakening-strengthening  
(d) linear slip weakening

- $\mu_s$  static and  $\mu_k$  kinetic coefficients of friction.

# Rate and state friction and regularization

- **Rate and state friction law**

- Rate  $v_t = |\underline{v}_t|$  – relative slip velocity
- State  $\theta$  –  $\approx$  internal time
- Dieterich–Ruina–Perrin (1979, 83, 95)

Frictional resistance

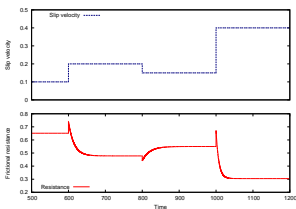
$$\sigma_t^c = |\sigma_n| [\mu_s + b\theta + a \ln(v_t/v_0)]$$

Evolution of the state variable

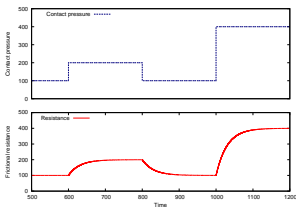
$$\dot{\theta} = -\frac{v_t}{L} \left[ \theta + \ln\left(\frac{v_t}{v_0}\right) \right]$$

- **Prakash-Clifton friction law (1992,2000)**

- Viscous type evolution of frictional resistance  $\sigma_t$
- $\dot{\sigma}_t = -\frac{v_t}{L} (\sigma_t + \mu\sigma_n)$



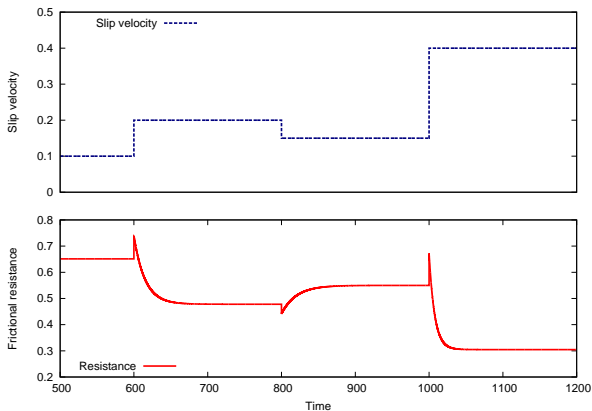
Rate and state friction law



Prakash-Clifton regularization

# Rate and state friction and regularization

- Rate and state friction law



# Rate and state friction and regularization

- **Rate and state friction law**

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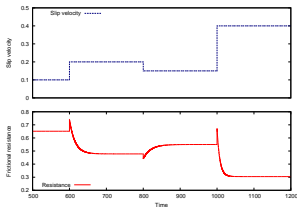
$$\sigma_t^c = |\sigma_n| [\mu_s + b\theta + a \ln(v_t/v_0)]$$

Evolution of the state variable

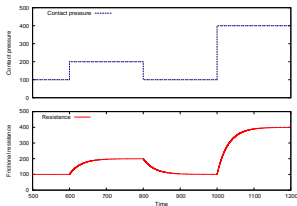
$$\dot{\theta} = -\frac{v_t}{L} \left[ \theta + \ln\left(\frac{v_t}{v_0}\right) \right]$$

- **Prakash-Clifton friction law (1992,2000)**

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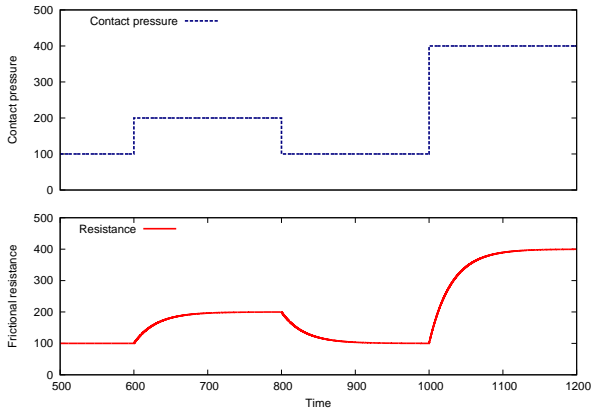
Rate and state friction law



Prakash-Clifton regularization

# Rate and state friction and regularization

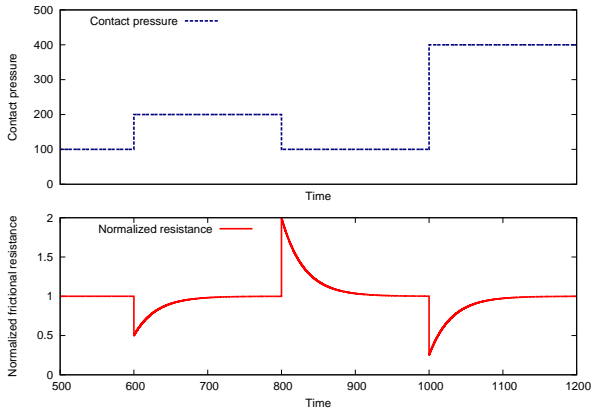
- Prakash-Clifton friction law (1992,2000)





# Rate and state friction and regularization

- Prakash-Clifton friction law (1992,2000)



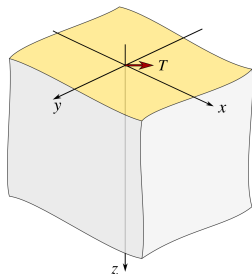
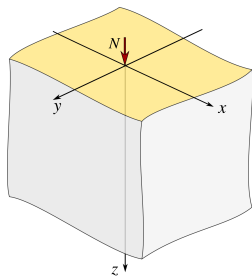
# Three-dimensional problem

- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem  
*concentrated normal force*
- Cerutti problem  
*concentrated tangential force*
- Displacements decay as  $\sim r^{-1}$

$$u_r(x, y, 0) = -\frac{1-2\nu}{4\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

$$u_z(x, y, 0) = \frac{1-\nu}{2\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

- Stress decay as  $\sim r^{-2}$
- Superposition principle



# Three-dimensional problem

## Normal force case

### ■ Full displacements:

$$u_x = \frac{N}{4\pi G} \left( \frac{xz}{r^3} - (1-2\nu) \frac{x}{r(r+z)} \right)$$

$$u_y = \frac{N}{4\pi G} \left( \frac{yz}{r^3} - (1-2\nu) \frac{y}{r(r+z)} \right)$$

$$u_z = \frac{N}{4\pi G} \left( \frac{z^2}{r^3} + \frac{2(1-\nu)}{r} \right)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

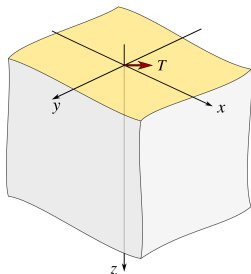
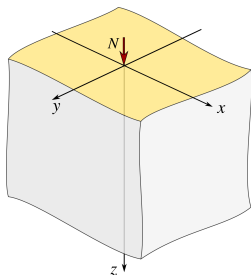
### ■ Stresses:

$$\sigma_x = \frac{N}{2\pi} \left[ \frac{1-2\nu}{\rho^2} \left( \left(1 - \frac{z}{r}\right) \frac{x^2 - y^2}{\rho^2} + \frac{zy^2}{r^3} \right) - \frac{3zx^2}{r^5} \right]$$

$$\sigma_y = \frac{N}{2\pi} \left[ \frac{1-2\nu}{\rho^2} \left( \left(1 - \frac{z}{r}\right) \frac{y^2 - x^2}{\rho^2} + \frac{zx^2}{r^3} \right) - \frac{3zy^2}{r^5} \right]$$

$$\sigma_z = -\frac{3N}{2\pi} \frac{z^3}{r^5}$$

$$\rho = \sqrt{x^2 + y^2}$$



# Three-dimensional problem

## Normal force case

- Full displacements:

$$u_x = \frac{N}{4\pi G} \left( \frac{xz}{r^3} - (1-2\nu) \frac{x}{r(r+z)} \right)$$

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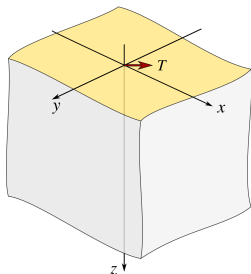
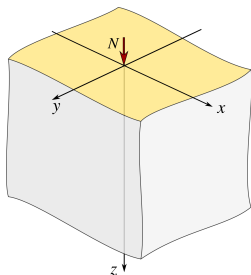
$$r = \sqrt{x^2 + y^2 + z^2}$$

- Stresses:

$$\sigma_{xy} = \frac{N}{2\pi} \left[ \frac{1-2\nu}{\rho^2} \left( \left\{ 1 - \frac{z}{r} \right\} \frac{xy}{\rho^2} - \frac{xyz}{r^3} \right) - \frac{3xyz}{r^5} \right]$$

$$\sigma_{xz} = -\frac{3N}{2\pi} \frac{xz^2}{r^5} \quad \sigma_{yz} = -\frac{3N}{2\pi} \frac{yz^2}{r^5}$$

$$\rho = \sqrt{x^2 + y^2}$$



- No friction, no adhesion

- Two elastic materials

$$E_1, \nu_1, E_2, \nu_2$$

- Effective elastic modulus

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

- Two parabolic surfaces

$$z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$$

- Solids of revolution or cylinders  $R_1, R_2$ , effective curvature radius:

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only for 3D!):

$$\delta = a^2 / R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$

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## Ueber die Berührung fester elastischer Körper.

(Von Herrn Heinrich Hertz.)

In der Theorie der Elasticität werden als Ursachen der Deformationen theils Kräfte, welche auf das Innere der Körper wirken, theils auf die Oberfläche wirkende Druckkräfte angenommen. Für beide Arten von Kräften kann der Fall eintreten, dass dieselben in einzelnen unendlich kleinen Theilen der Körper unendlich gross werden, so zwar, dass die Integrale der Kräfte über diese Theile genommen einen endlichen Werth behalten. Beschreiben wir alsdann um den Unstetigkeitspunkt eine geschlossene Fläche, deren Dimensionen sehr klein gegen die Dimensionen des ganzen Körpers sind, sehr gross hingegen im Vergleich zu den Dimensionen des Theils, in welchem die Kräfte angreifen, so können die Deformationen ausserhalb und innerhalb dieser Fläche ganz unabhängig von einander betrachtet werden. Ausserhalb hängen die Deformationen ab von der Gestalt des Gesamtkörpers, der Vertheilung der übrigen Kräfte und den endlichen Integralen der Kraftcomponenten im Unstetigkeitspunkte, innerhalb hingegen sie nur ab von der Vertheilung der im Innern selbst angreifenden Kräfte. Die Drucke und Deformationen im Innern sind gegen die im Aeussern unendlich gross.

Im Folgenden wollen wir einen hierher gehörigen Fall behandeln, der praktisches Interesse hat\*, den Fall nämlich, dass zwei elastische isotope Körper sich in einem sehr kleinen Theil ihrer Oberfläche berühren, und durch diesen Theil einen endlichen Druck der eine auf den andern ausüben. Die sich berührenden Oberflächen stellen wir uns als vollkommen glatt vor, d. h. wir nehmen nur einen senkrechten Druck zwischen den sich berührenden Theilen an. Das beiden Körpern nach der Deformation gemeinsame Stütz der Oberfläche wollen wir die Druckfläche, die Begrenzung

\* Vgl. Weisler, Die Lehre von der Elasticität und Festigkeit, Prag 1857; I. p. 43. Grashof, Theorie der Elasticität und Festigkeit, Berlin 1878; p. 49–54.

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Original paper by Heinrich Hertz “On the contact of elastic solids” (ENG trans.) (16 pages)

“His theory, worked out during the Christmas vacation 1880 at the age of 23(!), aroused considerable interest . . .” K.L. Johnson

# Hertzian contact

- No friction, no adhesion

- Two elastic materials

$$E_1, \nu_1, E_2, \nu_2$$

- Effective elastic modulus

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

- Two parabolic surfaces

$$z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$$

- Solids of revolution or cylinders  $R_1, R_2$ , effective curvature radius:

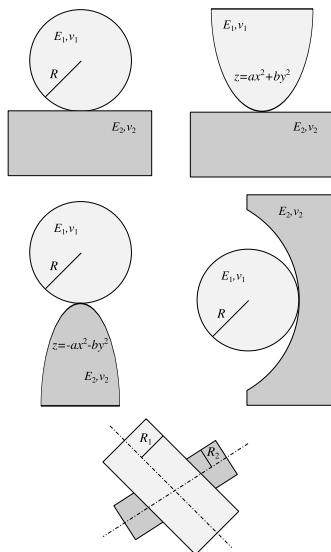
$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only for 3D!):

$$\delta = a^2 / R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$



Geometries resolved in the framework of Hertz theory

# Hertzian contact

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$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only for 3D!):

$$\delta = a^2/R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$

- Line contact (cylinders):

$$a = \left( \frac{4PR^*}{\pi E^*} \right)^{1/2}$$


$$p_0 = \frac{2P}{\pi a}$$

- Solids of revolution:

$$a = \left( \frac{3PR^*}{4E^*} \right)^{1/3}$$

$$p_0 = \frac{3P}{2\pi a^2}$$

# Classical contact problems

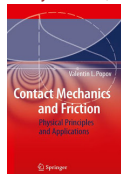
- Various problems with rigid flat stamps: *circular, elliptic, frictionless, full-stick, finite friction*
- Hertz theory  
*normal frictionless contact of elastic solids*   
 $E_i, \nu_i$  and  $z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1, 2$
- Wedges (*coin*) and cones
- Circular inclusion in a conforming hole  
Steuermann, 1939, Goodman, Keer, 1965
- Frictional indentation  $z \sim \chi^n$   
Incremental approach Mossakovski, 1954  
self-similar solution Spence, 1968, 1975
- Adhesive contact Johnson et al, 1971, 1976
- Contact with layered materials (coatings)
- Elastic-plastic and viscoelastic materials
- Sliding/rolling of non-conforming bodies  
Cattaneo (1938), Mindlin (1949), Galin (1953), Goryacheva (1998)  
Note:  $u_T \sim (1 - 2\nu)/G$ , so if  $(1 - 2\nu_1)/G_1 = (1 - 2\nu_2)/G_2$  tangential tractions do not change normal ones



K.L. Johnson (1985) J.R. Barber (2018)



I.G. Goryacheva (1998)



V.L. Popov (2017, 2nd Ed.)





Thank you for your attention!

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