Contact Mechanics and Elements of Tribology

Lecture 1.
Mechanical Contact

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Outline

Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_v = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \underline{u}\nabla)$$

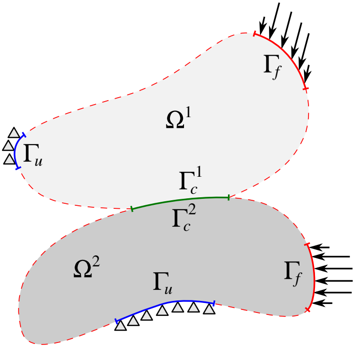
- Constitutive equation

$$\underline{\underline{\sigma}} = W'(\underline{\underline{\varepsilon}})$$

- Boundary conditions

$$\text{Dirichlet: } \underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$$

$$\text{Neumann: } \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$$



Two bodies in contact

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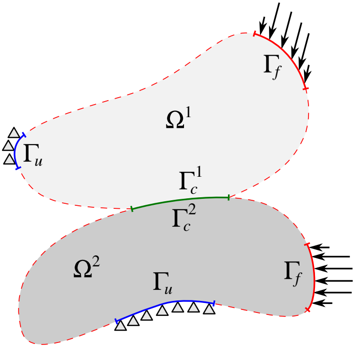


Two bodies in contact

- **Include contact conditions**

...

Intuitive conditions

1. No penetration

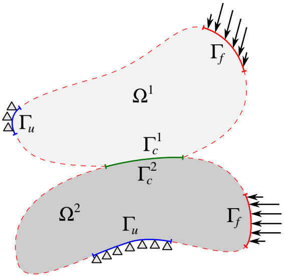
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

2. No adhesion

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \le 0, \forall \underline{x} \in \Gamma_c^i$$

3. No shear stress

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{\underline{I}} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$



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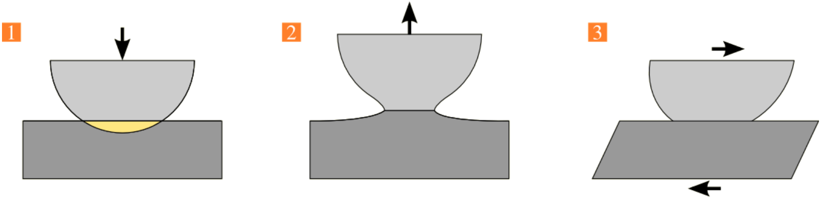

Intuitive contact conditions for frictionless and nonadhesive contact

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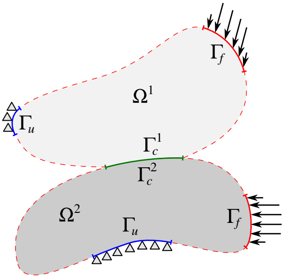

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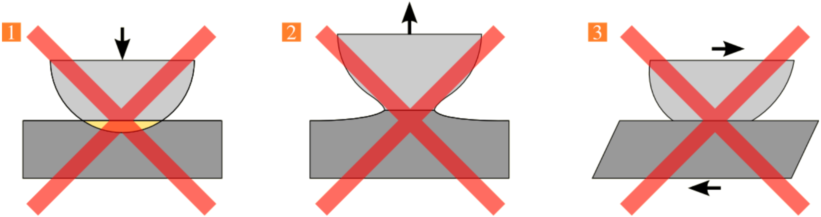

Intuitive contact conditions for frictionless and nonadhesive contact

Contact problem

≈ **Problem**

Find such contact pressure

$$p = -\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \geq 0$$

which being applied at Γ_c^1 and Γ_c^2 results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

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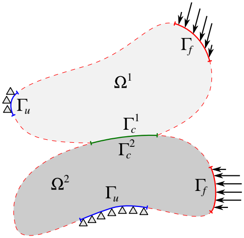

Two bodies in contact

- Unfortunately, we do not know Γ_c^1 in advance, it is also an unknown of the problem.
- **Related problem**
  Suppose that we know p on Γ_c

  Then what is the corresponding displacement field $\underline{u}$ in Ω^i? Or the other way around?

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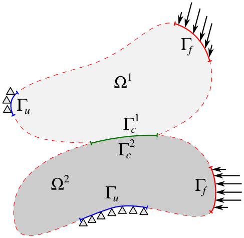

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 See Flamant problem TD

Concentrated forces in 2D

- Normal force (in-plane stresses and displacements (plane strain))

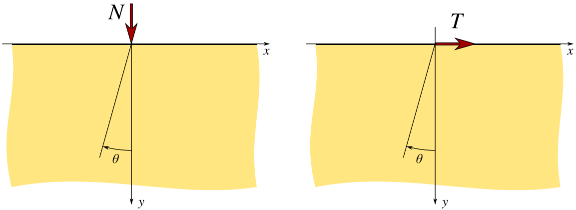
$$\sigma_r = -\frac{2N}{\pi}\frac{\cos(\theta)}{r} \quad \text{or} \quad \sigma_x = -\frac{2N}{\pi}\frac{x^2 y}{(x^2+y^2)^2},\ \sigma_y = -\frac{2N}{\pi}\frac{y^3}{(x^2+y^2)^2},\ \sigma_{xy} = -\frac{2N}{\pi}\frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N\cos(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\tan(\theta)\right] + C\cos(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} N\sin(\theta)\left[2(1-\nu)\ln(r) - 2\nu + (1-2\nu)(1-2\theta\,\mathrm{ctan}(\theta))\right] - C\sin(\theta)$$

- On the surface

$$u_x = -\frac{N(1+\nu)(1-2\nu)}{2E}\mathrm{sign}(x), \quad u_y = \frac{2N(1-\nu^2)}{\pi E}\log(|x|) + C$$



Concentrated forces in 2D

- Tangential force (in-plane stresses and displacements (plane strain))

$$\sigma_r = \frac{2T}{\pi}\frac{\sin(\theta)}{r} \quad \text{or} \quad \sigma_x = -\frac{2T}{\pi}\frac{x^3}{(x^2+y^2)^2},\; \sigma_y = -\frac{2T}{\pi}\frac{xy^2}{(x^2+y^2)^2},\; \sigma_{xy} = -\frac{2T}{\pi}\frac{x^2y}{(x^2+y^2)^2}$$

$$u_r = -\frac{1+\nu}{\pi E}T\sin(\theta)\left[2(1-\nu)\ln(r) - (1-2\nu)\theta\mathrm{ctan}(\theta)\right] - C\sin(\theta)$$
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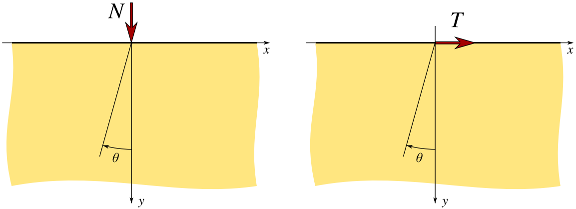
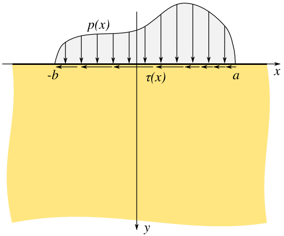


Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements



Tractions on the surface

$$\sigma_x(x,y) = -\frac{2y}{\pi}\int\limits_{-b}^{a}\frac{p(s)(x-s)^2\,ds}{((x-s)^2+y^2)^2} - \frac{2}{\pi}\int\limits_{-b}^{a}\frac{\tau(s)(x-s)^3\,ds}{((x-s)^2+y^2)^2}$$

$$\sigma_y(x,y) = -\frac{2y^3}{\pi}\int\limits_{-b}^{a}\frac{p(s)\,ds}{((x-s)^2+y^2)^2} - \frac{2y^2}{\pi}\int\limits_{-b}^{a}\frac{\tau(s)(x-s)\,ds}{((x-s)^2+y^2)^2}$$

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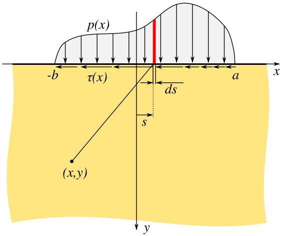

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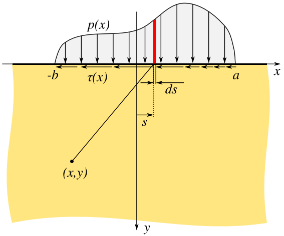

Tractions on the surface

$$u_x(x,0) = -\mathrm{sign}(x)\frac{(1-2\nu)(1+\nu)}{2E}\left[\int\limits_{-b}^{x} p(s)\,ds - \int\limits_{x}^{a} p(s)\,ds\right] - \frac{2(1-\nu^2)}{\pi E}\int\limits_{-b}^{a}\tau(s)\ln|x-s|\,ds + C_1$$

Distributed load

- Distributed tractions $p(x)dx = dN(x)$, $\tau(x)dx = dT(x)$
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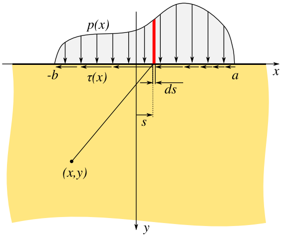
- Or rather their derivatives along the surface



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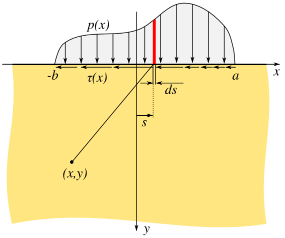

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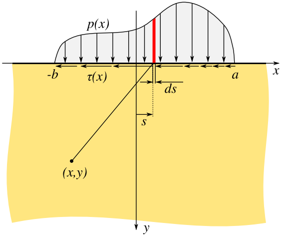

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Inverse problem

- Link displacement derivatives with tractions

$$\int_{-b}^{a} \frac{\tau(s)}{x-s}\,ds = -\frac{\pi(1-2\nu)}{2(1-\nu)}p(x) - \frac{\pi E}{2(1-\nu^2)}u_{x,x}(x,0)$$

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- If in contact interface we can prescribe $p, u_{x,x}$ or $\tau, u_{y,x}$, then the problem reduces to

$$\int_{-b}^{a} \frac{\mathcal{F}(s)}{x-s}ds = \mathcal{U}(x)$$

- The general solution (case $a = b$):

$$\mathcal{F}(x) = \frac{1}{\pi^2\sqrt{a^2-x^2}}\int_{-a}^{a}\frac{\sqrt{a^2-s^2}\,\mathcal{U}(s)\,ds}{x-s} + \frac{C}{\pi\sqrt{a^2-x^2}}, \quad C = \int_{-a}^{a}\mathcal{F}(s)ds$$

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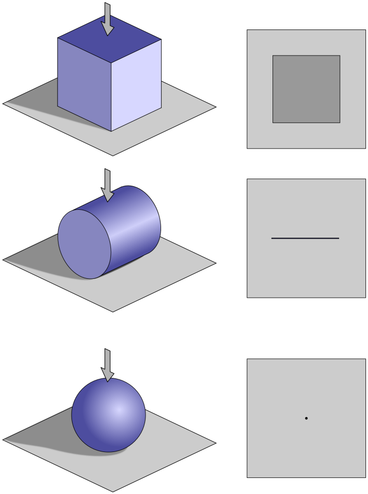
flat frictionless punch, consider P.V.

Types of contact

- Known contact zone
    - conformal geometry
      flat-to-flat, cylinder in a hole
    - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone
  general case
- Point and line contact
- Frictionless
  conservative, energy minimization problem
- Frictional
  path-dependent solution, from the first touch to the current moment

Example



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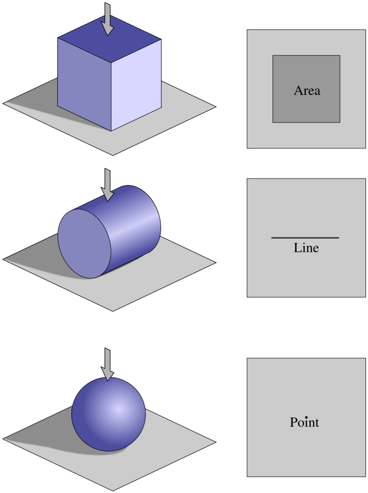
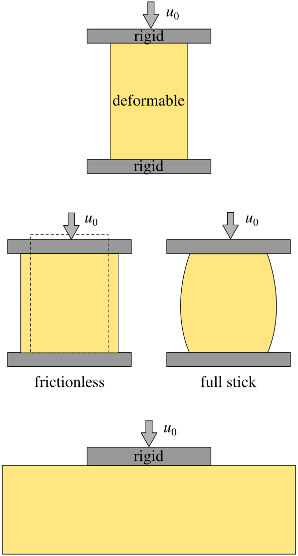


Analogy with boundary conditions

Flat geometry

- Compression of a cylinder
- Frictionless $u_z = u_0$
- Full stick conditions $\underline{u} = u_0 \underline{e}_z$
- Rigid flat indenter $u_z = u_0$



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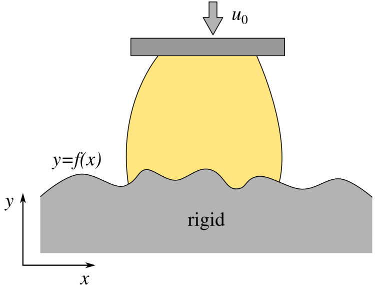
Curved geometry

- Polar/spherical coordinates
  $u_r = u_0$
- If frictionless contact on rigid surface $y = f(x)$ is retained by high pressure

$$(\underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}) \cdot \underline{\boldsymbol{e}}_y = f((\underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}) \cdot \underline{\boldsymbol{e}}_x)$$



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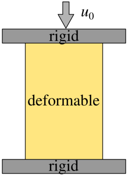

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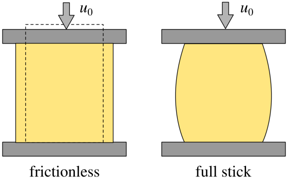
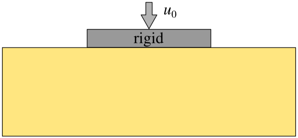


Transition to finite friction

-  ≈ From full stick, decrease f by keeping $u_z = 0$ and by replacing in-plane Dirichlet BC by in-plane Neumann BC



Analogy with boundary conditions II

In general

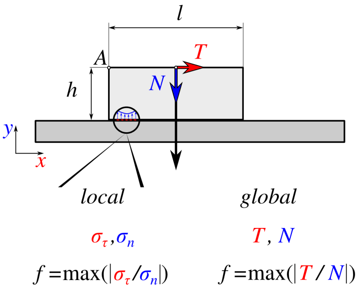
- Type I: prescribed tractions 
  $p(x,y), \tau_x(x,y), \tau_y(x,y)$
- Type II: prescribed displacements
  $\underline{\boldsymbol{u}}(x,y)$
- Type III: tractions and displacements
  $u_z(x,y), \tau_x(x,y), \tau_y(x,y)$ or
  $p(x,y), u_x(x,y), u_y(x,y)$
- Type IV: displacements and relation between tractions
  $u_z(x,y), \tau_x(x,y) = \pm f p(x,y)$

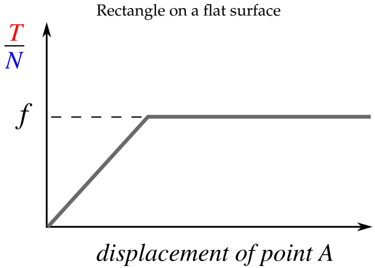
To be continued. . .

- Existence of frictional resistance is evident
- Independence of the nominal contact area

Think about adhesion and introduce a threshold in the interface τ_c

- Globally:
  - stick: $T < T_c(N)$
  - slip: $T = T_c(N)$
- From experiments:
  - Threshold $T_c \sim N$
  - Friction coefficient $f = |T_c/N|$
- Locally
  - stick: $\sigma_\tau < \tau_c(\sigma_n)$
  - slip: $\sigma_\tau = f\sigma_n$





Evidence of friction

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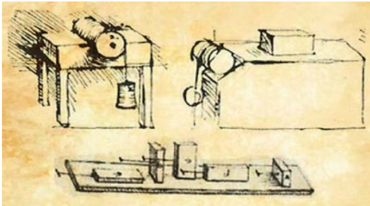
  - stick: $\sigma_\tau < \tau_c(\sigma_n)$

  - slip: $\sigma_\tau = f\sigma_n$
- Torque



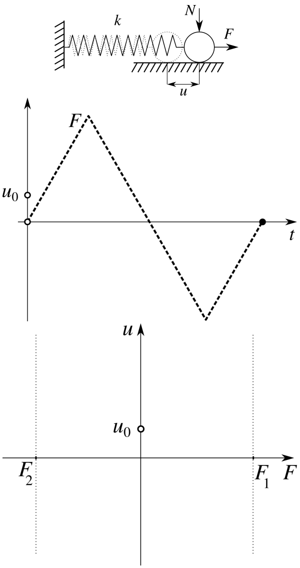
First ever frictional experiments: notebook drawings of Leonardo da Vinci

- proportionality between weight and frictional force
- friction is independent on the contact area

Friction is not that intuitive

- Frictionless
  conservative, energy minimization problem
- Frictional
  path-dependent solution, from the first touch to the current moment

Example



Friction is not that intuitive

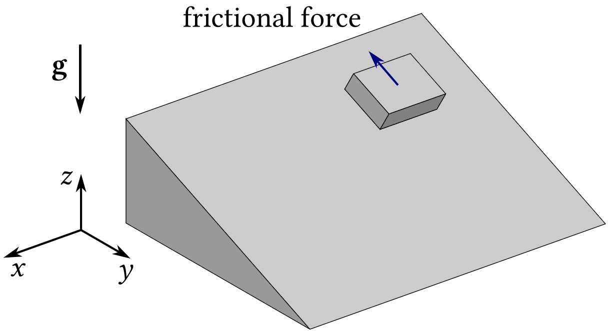
What does your intuition says?

https://forms.gle/h4QxwzzztEW2iTTL8

When the force starts to decrease, the contact point
□ goes to the left □ does not move □ goes to the right

Direction of sliding

Sliding orthogonally to the slope of the inclined plane



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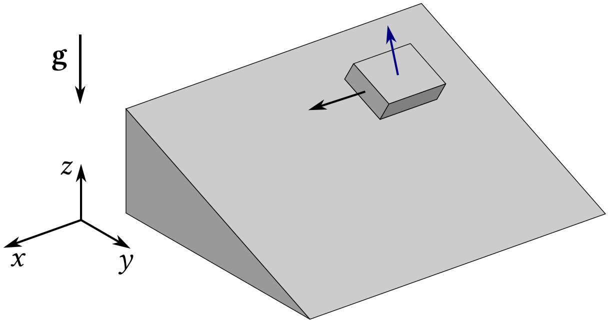

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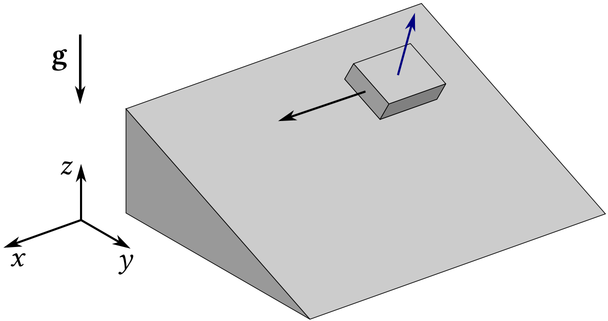
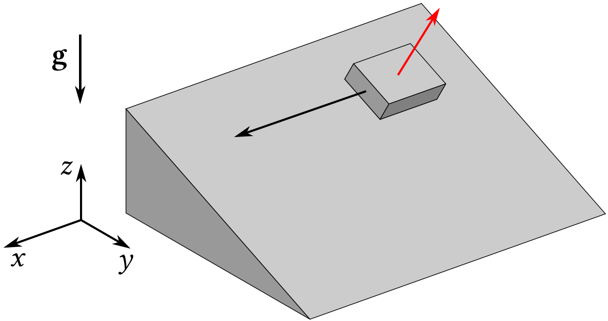
Sliding orthogonally to the slope of the inclined plane



Sliding orthogonally to the slope of the inclined plane



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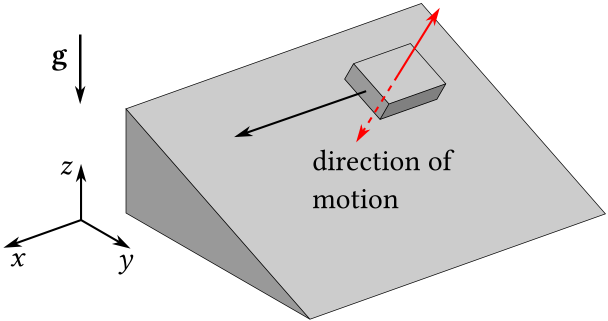
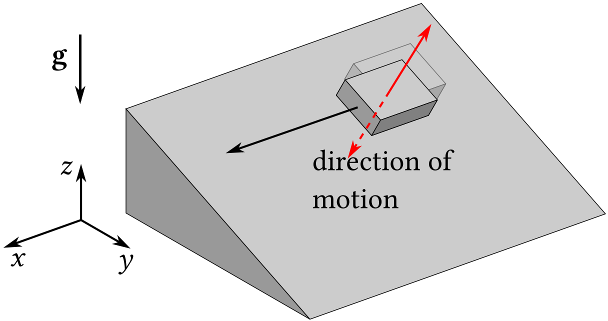


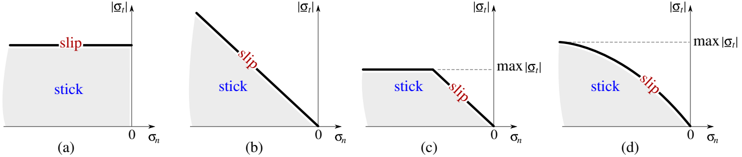
Direction of sliding

Sliding orthogonally to the slope of the inclined plane



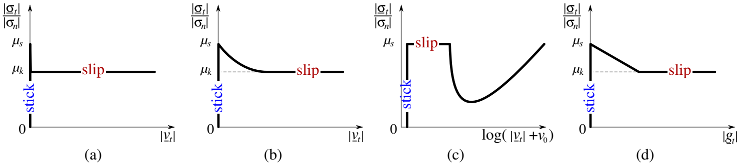
More friction laws

- **Static criteria**



(a) Tresca (b) Amontons-Coulomb (c) Coulomb-Orowan (d) Shaw

- **Kinetic criteria**



(a,b) velocity weakening (c) velocity weakening-strengthening
(d) linear slip weakening

- μ_s static and μ_k kinetic coefficients of friction.

Rate and state friction and regularization

- **Rate and state friction law**
  - Rate $v_t = |\underline{v}_t|$ – relative slip velocity
  - State θ – $\approx$ internal time
  - Dieterich–Ruina–Perrin (1979, 83, 95)

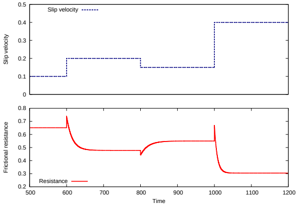
    Frictional resistance

    $$\sigma_t^c = |\sigma_n| \left[\mu_s + b\theta + a\ln(v_t/v_0)\right]$$

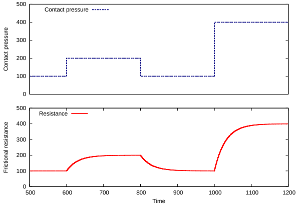
    Evolution of the state variable

    $$\dot\theta = -\frac{v_t}{L}\left[\theta + \ln\left(\frac{v_t}{v_0}\right)\right]$$

- **Prakash-Clifton friction law (1992,2000)**
  - Viscous type evolution of frictional resistance σ_t
  - $\dot\sigma_t = -\frac{v_t}{L}(\sigma_t + \mu\sigma_n)$

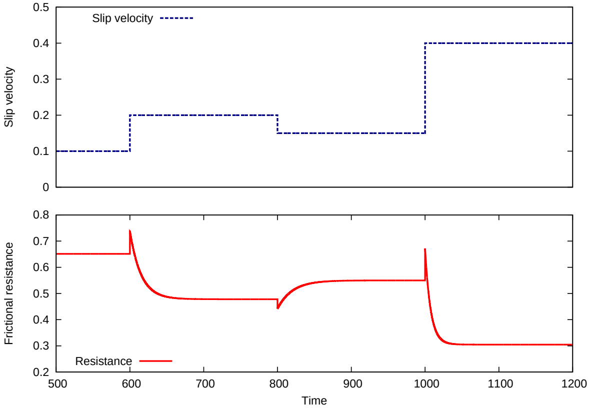


Rate and state friction law



Prakash-Clifton regularization

- **Rate and state friction law**



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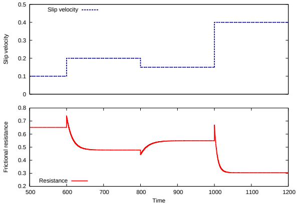

Rate and state friction law

- **Prakash-Clifton friction law (1992,2000)**
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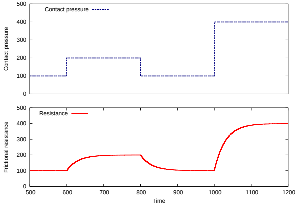
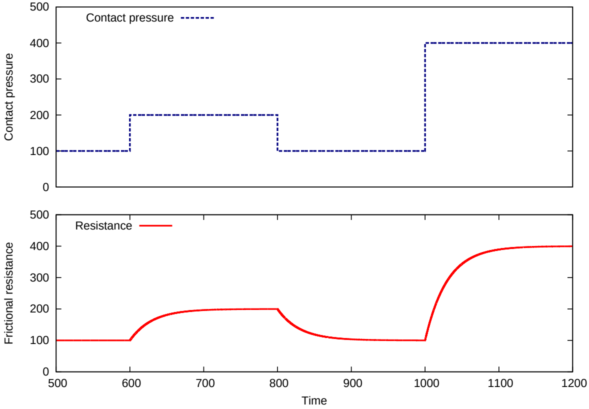


Prakash-Clifton regularization

Rate and state friction and regularization

- **Prakash-Clifton friction law (1992,2000)**



Rate and state friction and regularization

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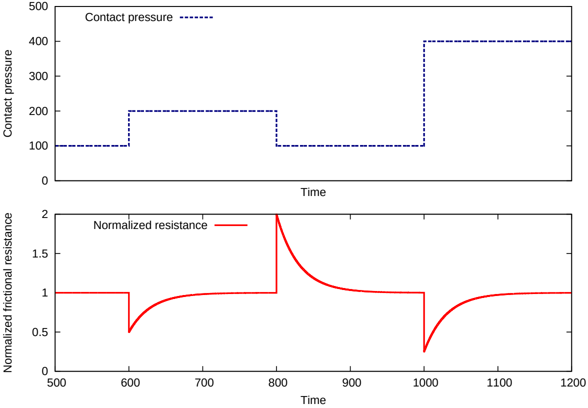


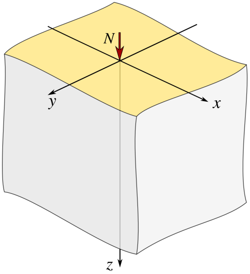
Three-dimensional problem

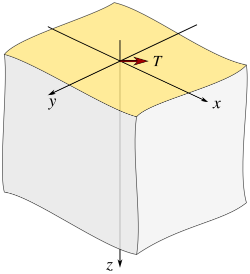
- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem
  concentrated normal force
- Cerutti problem
  concentrated tangential force
- Displacements decay as $\sim r^{-1}$

$$u_r(x,y,0) = -\frac{1-2\nu}{4\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

$$u_z(x,y,0) = \frac{1-\nu}{2\pi G}\frac{N}{\sqrt{x^2+y^2}}$$

- Stress decay as $\sim r^{-2}$
- Superposition principle





Three-dimensional problem

Normal force case

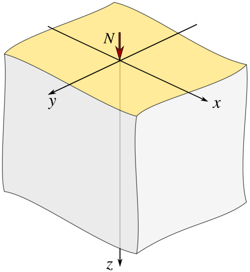
- Full displacements:

$$u_x = \frac{N}{4\pi G}\left(\frac{xz}{r^3} - (1-2\nu)\frac{x}{r(r+z)}\right)$$

$$u_x = \frac{N}{4\pi G}\left(\frac{yz}{r^3} - (1-2\nu)\frac{y}{r(r+z)}\right)$$

$$u_z = \frac{N}{4\pi G}\left(\frac{z^2}{r^3} + \frac{2(1-\nu)}{r}\right)$$

$$r = \sqrt{x^2+y^2+z^2}$$



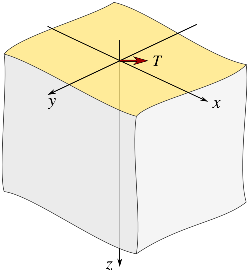
- Stresses:

$$\sigma_x = \frac{N}{2\pi}\left[\frac{1-2\nu}{\rho^2}\left(\left\{1-\frac{z}{r}\right\}\frac{x^2-y^2}{\rho^2} + \frac{zy^2}{r^3}\right) - \frac{3zx^2}{r^5}\right]$$

$$\sigma_y = \frac{N}{2\pi}\left[\frac{1-2\nu}{\rho^2}\left(\left\{1-\frac{z}{r}\right\}\frac{y^2-x^2}{\rho^2} + \frac{zx^2}{r^3}\right) - \frac{3zy^2}{r^5}\right]$$

$$\sigma_z = -\frac{3N}{2\pi}\frac{z^3}{r^5}$$

$$\rho = \sqrt{x^2+y^2}$$



Normal force case

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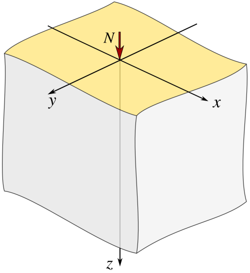
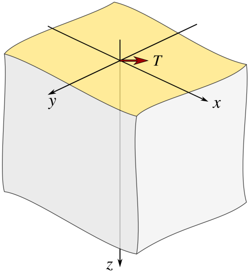
- Stresses:

$$\sigma_{xy} = \frac{N}{2\pi}\left[\frac{1-2\nu}{\rho^2}\left(\left\{1-\frac{z}{r}\right\}\frac{xy}{\rho^2} - \frac{xyz}{r^3}\right) - \frac{3xyz}{r^5}\right]$$

$$\sigma_{xz} = -\frac{3N}{2\pi}\frac{xz^2}{r^5} \qquad \sigma_{yz} = -\frac{3N}{2\pi}\frac{yz^2}{r^5}$$

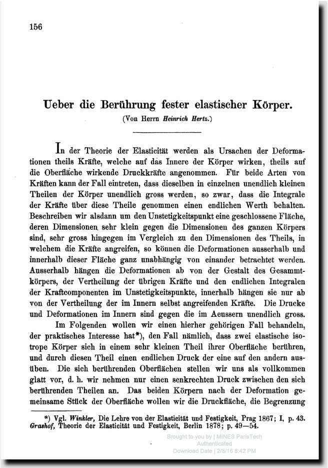
$$\rho = \sqrt{x^2+y^2}$$





Hertzian contact

- No friction, no adhesion
- Two elastic materials
  E_1, ν_1, E_2, ν_2
- Effective elastic modulus
  $$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$
- Two parabolic surfaces
  $z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$
- Solids of revolution or cylinders R_1,R_2, effective curvature radius:
  $$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$
- Displacement and contact radius (only for 3D!):
  $$\delta = a^2/R^*$$
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}}, \quad |r| < a$$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Ueber die Berührung fester elastischer Körper.

(Von Herrn *Heinrich Hertz.*)

In der Theorie der Elasticität werden als Ursachen der Deformationen theils Kräfte, welche auf das Innere der Körper wirken, theils auf die Oberfläche wirkende Druckkräfte angenommen. Für beide Arten von Kräften kann der Fall eintreten, dass dieselben in einzelnen unendlich kleinen Theilen der Körper unendlich gross werden, so zwar, dass die Integrale der Kräfte über diese Theile genommen einen endlichen Werth behalten. Beschreiben wir alsdann um den Unstetigkeitspunkt eine geschlossene Fläche, deren Dimensionen sehr klein gegen die Dimensionen des ganzen Körpers sind, sehr gross hingegen im Vergleich zu den Dimensionen des Theils, in welchem die Kräfte angreifen, so können die Deformationen ausserhalb und innerhalb dieser Fläche ganz unabhängig von einander betrachtet werden. Ausserhalb hängen die Deformationen ab von der Gestalt des Gesammtkörpers, der Vertheilung der übrigen Kräfte und den endlichen Integralen der Kraftcomponenten im Unstetigkeitspunkte, innerhalb hängen sie nur ab von der Vertheilung der im Innern selbst angreifenden Kräfte. Die Drucke und Deformationen im Innern sind gegen die im Aeussern unendlich gross.

Im Folgenden wollen wir einen hierher gehörigen Fall behandeln, der praktisches Interesse hat*), den Fall nämlich, dass zwei elastische isotrope Körper sich in einem sehr kleinen Theil ihrer Oberfläche berühren, und durch diesen Theil einen endlichen Druck der eine auf den andern ausüben. Die sich berührenden Oberflächen stellen wir uns als vollkommen glatt vor, d. h. wir nehmen nur einen senkrechten Druck zwischen den sich berührenden Theilen an. Das beiden Körpern nach der Deformation gemeinsame Stück der Oberfläche wollen wir die Druckfläche, die Begrenzung

*) Vgl. *Winkler*, Die Lehre von der Elasticität und Festigkeit, Prag 1867; I, p. 43. *Grashof*, Theorie der Elasticität und Festigkeit, Berlin 1878; p. 49—54.

Original paper by Henrich Hertz “On the contact of elastic solids” (ENG trans.) (16 pages)

“His theory, worked out during the Christmas vacation 1880 at the age of 23(!), aroused considerable interest . . . ” K.L. Johnson

Hertzian contact

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- Two elastic materials
  E_1, ν_1, E_2, ν_2
- Effective elastic modulus
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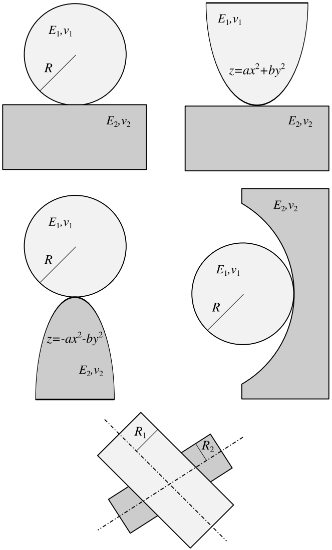
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}},\ |r| < a$$



Geometries resolved in the framework of Hertz theory

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  E_1, ν_1, E_2, ν_2
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- Displacement and contact radius (only for 3D!):
  $\delta = a^2/R^*$
- Contact pressure:
  $$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}},\ |r| < a$$
- Line contact (cylinders):
  $$a = \left(\frac{4PR^*}{\pi E^*}\right)^{1/2}$$
  $$p_0 = \frac{2P}{\pi a}$$
- Solids of revolution:
  $$a = \left(\frac{3PR^*}{4E^*}\right)^{1/3}$$
  $$p_0 = \frac{3P}{2\pi a^2}$$

Classical contact problems

- Various problems with rigid flat stamps: *circular, elliptic, frictionless, full-stick, finite friction*
- Hertz theory

  normal frictionless contact of elastic solids 

  E_i, ν_i and $z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1, 2$
- Wedges *(coin)* and cones
- Circular inclusion in a conforming hole

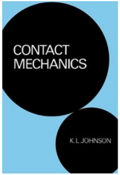
  Steuermann, 1939,Goodman, Keer, 1965
- Frictional indentation $z \sim x^n$

  Incremental approach Mossakovski, 1954

  self-similar solution Spence, 1968, 1975
- Adhesive contact Johnson et al, 1971, 1976
- Contact with layered materials (coatings)
- Elastic-plastic and viscoelastic materials
- Sliding/rolling of non-conforming bodies

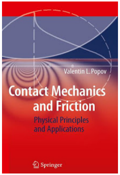
  Cattaneo (1938), Mindlin (1949), Galin (1953), Goryacheva (1998)

  Note: $u_r \sim (1-2\nu)/G$, so if $(1-2\nu_1)/G_1 = (1-2\nu_2)/G_2$ tangential tractions do not change normal ones



K.L. Johnson (1985) J.R. Barber (2018)

I.G. Goryacheva (1998)



V.L. Popov (2017, 2nd Ed.)

Thank you for your attention!