

# Contact Mechanics and Elements of Tribology

## Lecture 1. *Mechanical Contact*

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@ Centre des Matériaux (& virtually)  
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## Part I

- 1 Balance equations
- 2 Intuitive notions
- 3 Formalization of frictionless contact
- 4 Flamant's solution
- 5 Integration of Flamant' solution
- 6 Displacements and tractions
- 7 Contact types
- 8 Analogy with boundary conditions

## Part II

- 1 Evidence of friction
- 2 Friction models
- 3 Boussinesq, Cerruti
- 4 Hertzian contact
- 5 Classical contact problems

# Boundary value problem in elasticity

- Reference and current configurations

$$\underline{x} = \underline{X} + \underline{u}$$

- Balance equation (strong form)

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{f}_{-v} = 0, \forall \underline{x} \in \Omega^i$$

- Displacement compatibility

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \underline{u} \nabla)$$

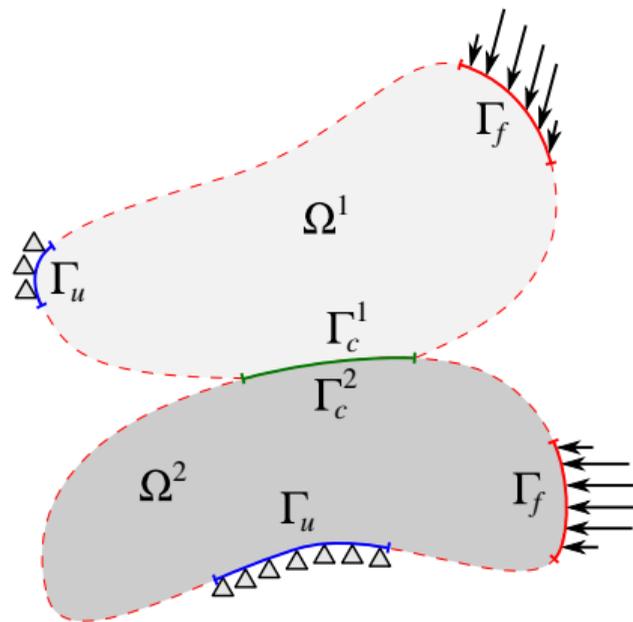
- Constitutive equation

$$\underline{\underline{\sigma}} = W'(\underline{\underline{\varepsilon}})$$

- Boundary conditions

$$\text{Dirichlet: } \underline{u} = \underline{u}^0, \forall \underline{x} \in \Gamma_u$$

$$\text{Neumann: } \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t}^0, \forall \underline{x} \in \Gamma_f$$



Two bodies in contact

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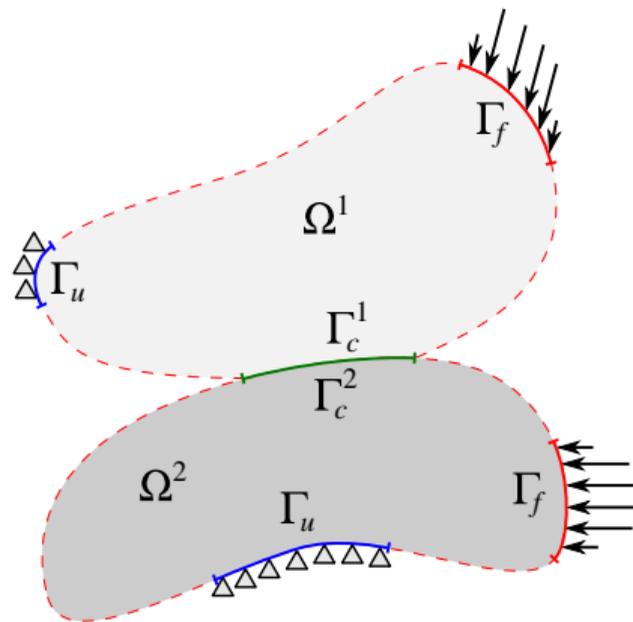
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Two bodies in contact

- **Include contact conditions**

# Intuitive conditions

- 1 No penetration

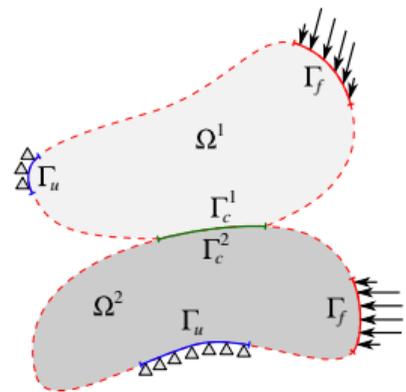
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

- 2 No adhesion

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \leq 0, \forall \underline{x} \in \Gamma_c^i$$

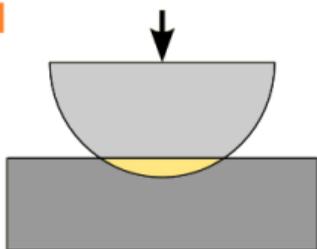
- 3 No shear stress

$$\underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n}) = 0, \forall \underline{x} \in \Gamma_c^i$$

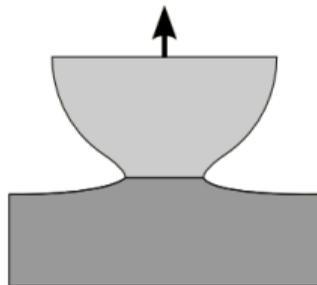


Two bodies in contact

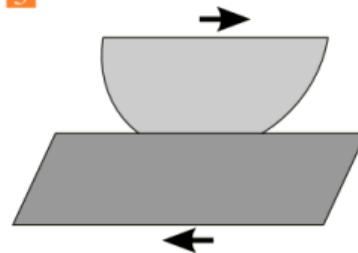
1



2



3



Intuitive contact conditions for frictionless and nonadhesive contact

# Intuitive conditions

- 1 No penetration

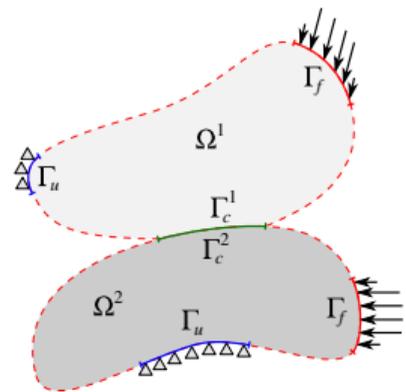
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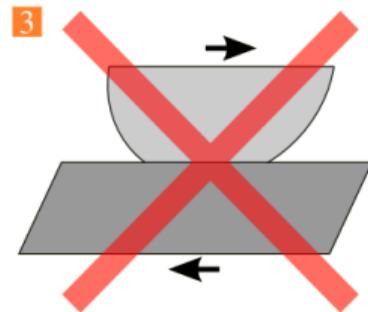
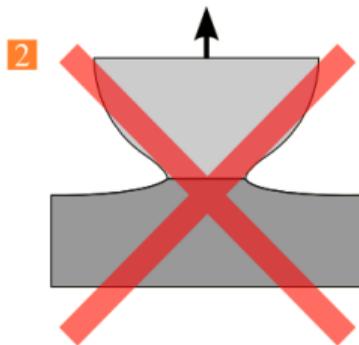
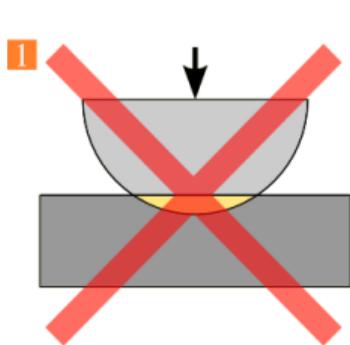
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Two bodies in contact



Intuitive contact conditions for frictionless and nonadhesive contact

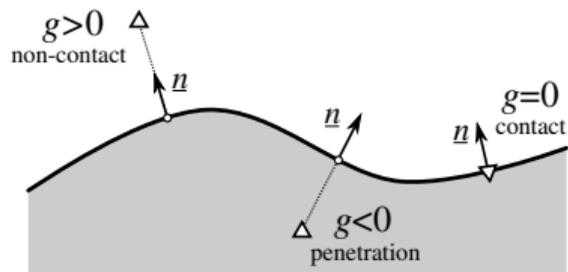
# Gap function

## ■ Gap function $g$

- gap = - penetration
- asymmetric function
- defined for
  - separation  $g > 0$
  - contact  $g = 0$
  - penetration  $g < 0$
- governs normal contact

## ■ Master and slave split

*Gap function is determined for all slave points with respect to the master surface*



*Gap between a slave point and a master surface*

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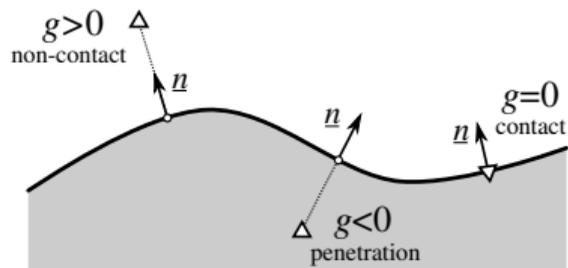
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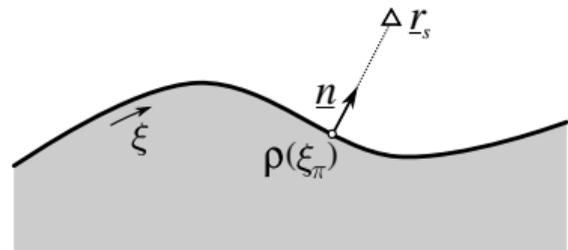
## ■ Normal gap

$$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)],$$

$\underline{n}$  is a unit normal vector,  $\underline{r}_s$  slave point,  $\underline{\rho}(\xi_\pi)$  projection point at master surface



*Gap between a slave point and a master surface*



*Definition of the normal gap*

# Frictionless or normal contact conditions

- **No penetration**

*Always non-negative gap*

$$g \geq 0$$

- **No adhesion**

*Always non-positive contact pressure*

$$\sigma_n^* \leq 0$$

- **Complementary condition**

*Either zero gap and non-zero pressure, or non-zero gap and zero pressure*

$$g \sigma_n = 0$$

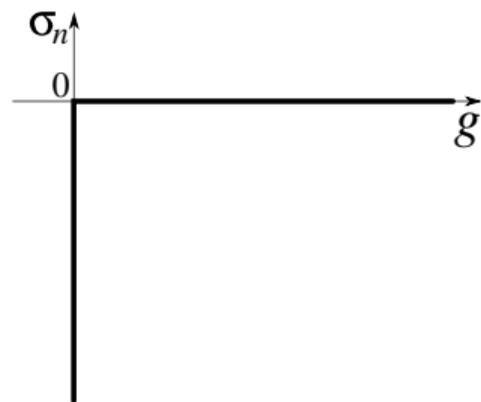
- **No shear transfer (automatically)**

$$\underline{\sigma}_t^{**} = 0$$

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$$\sigma_n^* = (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n} = \underline{\underline{\sigma}} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^{**} = \underline{\underline{\sigma}} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\underline{\sigma}} \cdot (\underline{I} - \underline{n} \otimes \underline{n})$$



Scheme explaining normal contact conditions

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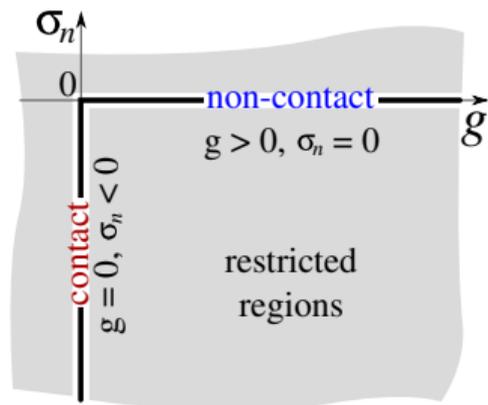
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**Improved scheme explaining normal contact conditions**

# Frictionless or normal contact conditions

In mechanics:

*Normal contact conditions*

≡

*Frictionless contact conditions*

≡

*Hertz<sup>1</sup>-Signorini<sup>[2]</sup> conditions*

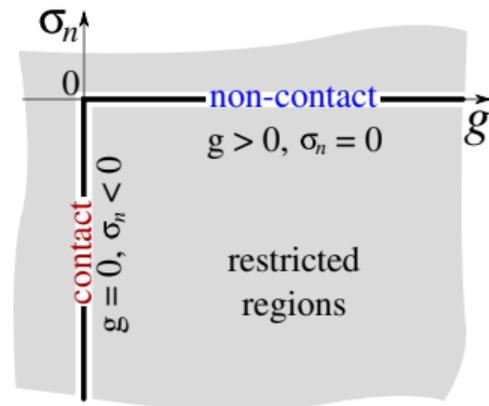
≡

*Hertz<sup>1</sup>-Signorini<sup>[2]</sup>-Moreau<sup>[3]</sup> conditions*

also known in **optimization theory** as

*Karush<sup>[4]</sup>-Kuhn<sup>[5]</sup>-Tucker<sup>[6]</sup> conditions*

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$



**Improved scheme explaining normal contact conditions**

<sup>1</sup>Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

<sup>2</sup>Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

<sup>3</sup>Jean Jacques Moreau (1923) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

<sup>4</sup>William Karush (1917–1997), <sup>5</sup>Harold William Kuhn (1925) American mathematicians,

<sup>6</sup>Albert William Tucker (1905–1995) a Canadian mathematician.

# Contact problem

## ≈ Problem

Find such contact pressure

$$p = -\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \geq 0$$

which being applied at  $\Gamma_c^1$  and  $\Gamma_c^2$  results in

$$\underline{x}^1 = \underline{x}^2, \forall \underline{x}^1 \in \Gamma_c^1, \underline{x}^2 \in \Gamma_c^2$$

and evidently

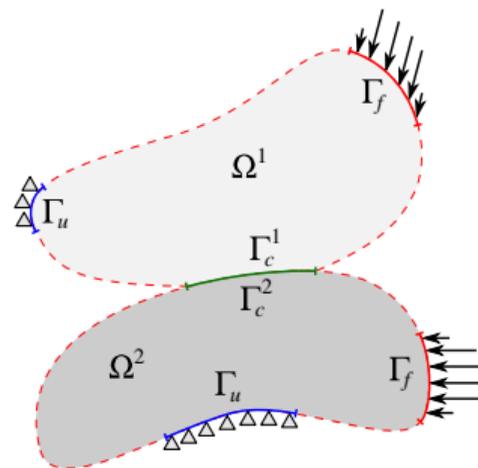
$$\Omega^1(t) \cap \Omega^2(t) = \emptyset$$

- Unfortunately, we do not know  $\Gamma_c^1$  in advance, it is also an unknown of the problem.

## ■ Related problem

Suppose that we know  $p$  on  $\Gamma_c$

Then what is the corresponding displacement field  $\underline{u}$  in  $\Omega^i$ ? Or the other way around?



Two bodies in contact

# Contact problem

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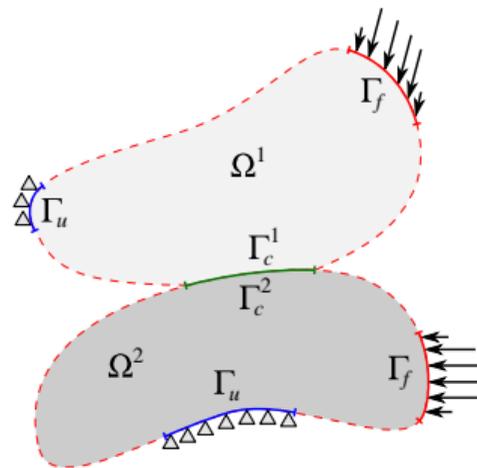
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See Flamant problem TD



Two bodies in contact

# Concentrated forces in 2D

- Normal force (in-plane stresses and displacements (plane strain))

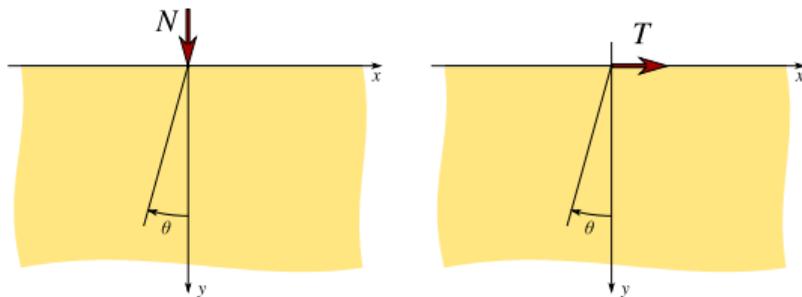
$$\sigma_r = -\frac{2N}{\pi} \frac{\cos(\theta)}{r} \quad \text{or} \quad \sigma_x = -\frac{2N}{\pi} \frac{x^2 y}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2N}{\pi} \frac{y^3}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2N}{\pi} \frac{xy^2}{(x^2+y^2)^2}$$

$$u_r = \frac{1+\nu}{\pi E} N \cos(\theta) [2(1-\nu) \ln(r) - (1-2\nu)\theta \tan(\theta)] + C \cos(\theta)$$

$$u_\theta = \frac{1+\nu}{\pi E} N \sin(\theta) [2(1-\nu) \ln(r) - 2\nu + (1-2\nu)(1-2\theta \cotan(\theta))] - C \sin(\theta)$$

- On the surface

$$u_x = -\frac{N(1+\nu)(1-2\nu)}{2E} \text{sign}(x), \quad u_y = \frac{2N(1-\nu^2)}{\pi E} \log(|x|) + C$$



# Concentrated forces in 2D

- Tangential force (in-plane stresses and displacements (plane strain))

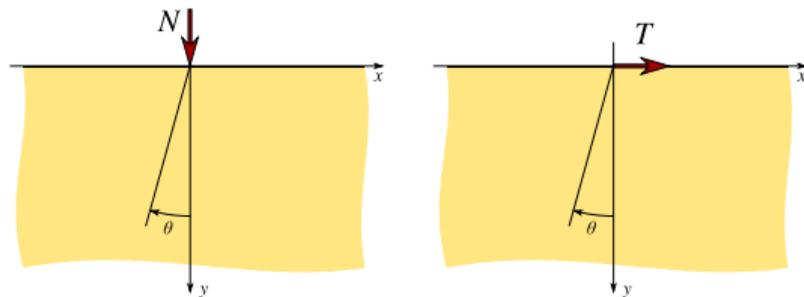
$$\sigma_r = \frac{2T}{\pi} \frac{\sin(\theta)}{r} \quad \text{or} \quad \sigma_x = -\frac{2T}{\pi} \frac{x^3}{(x^2+y^2)^2}, \quad \sigma_y = -\frac{2T}{\pi} \frac{xy^2}{(x^2+y^2)^2}, \quad \sigma_{xy} = -\frac{2T}{\pi} \frac{x^2y}{(x^2+y^2)^2}$$

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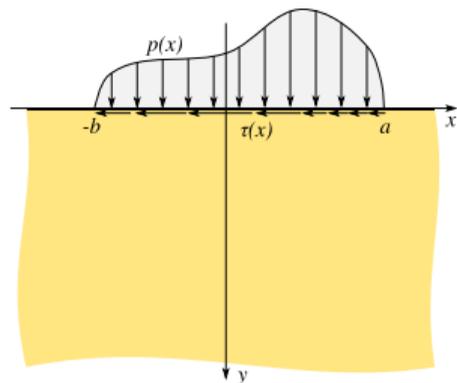
# Distributed load

- Distributed tractions  $p(x)dx = dN(x)$ ,  $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements

$$\sigma_x(x, y) = -\frac{2y}{\pi} \int_{-b}^a \frac{p(s)(x-s)^2 ds}{((x-s)^2 + y^2)^2} - \frac{2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s)^3 ds}{((x-s)^2 + y^2)^2}$$

$$\sigma_y(x, y) = -\frac{2y^3}{\pi} \int_{-b}^a \frac{p(s) ds}{((x-s)^2 + y^2)^2} - \frac{2y^2}{\pi} \int_{-b}^a \frac{\tau(s)(x-s) ds}{((x-s)^2 + y^2)^2}$$

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*Tractions on the surface*

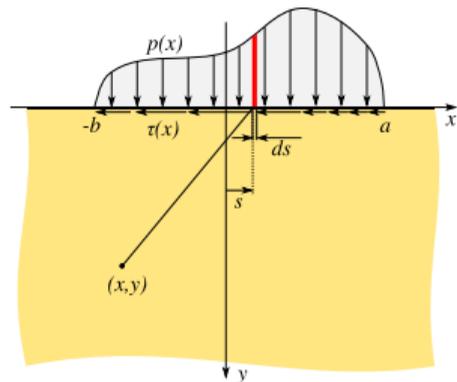
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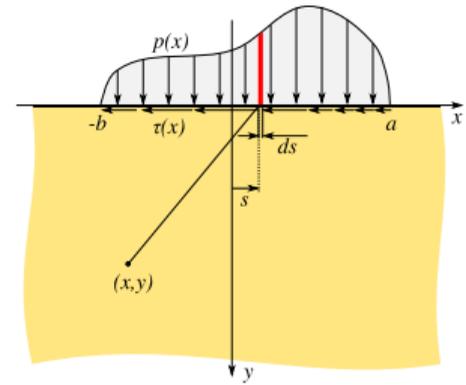
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# Distributed load

- Distributed tractions  $p(x)dx = dN(x)$ ,  $\tau(x)dx = dT(x)$
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- Consider displacements on the surface

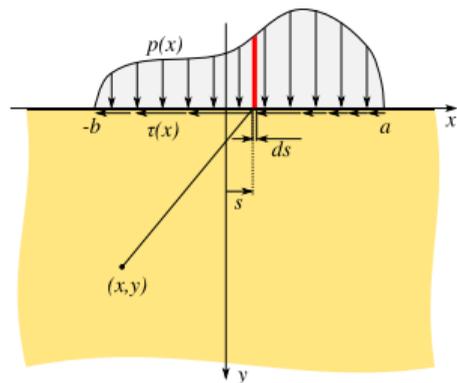


*Tractions on the surface*

$$u_x(x, 0) = -\text{sign}(x) \frac{(1 - 2\nu)(1 + \nu)}{2E} \left[ \int_{-b}^x p(s) ds - \int_x^a p(s) ds \right] - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a \tau(s) \ln|x - s| ds + C_1$$

# Distributed load

- Distributed tractions  $p(x)dx = dN(x)$ ,  $\tau(x)dx = dT(x)$
- Use superposition principle for the stress state and for displacements
- Consider displacements on the surface
- Or rather their derivatives along the surface



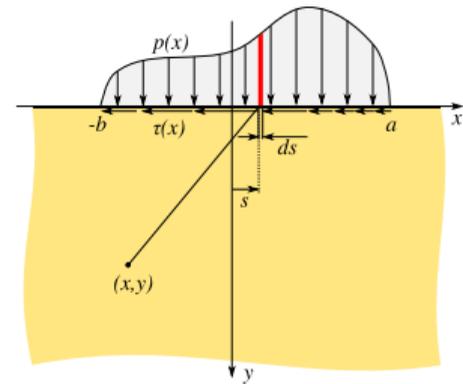
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$$u_{x,x}(x, 0) = -\text{sign}(x) \frac{(1-2\nu)(1+\nu)}{E} p(x) - \frac{2(1-\nu^2)}{\pi E} \int_{-b}^a \frac{\tau(s)}{x-s} ds$$

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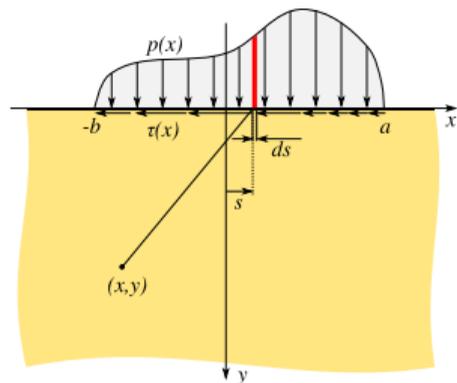


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*Traction on the surface*

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$$u_{y,x}(x, 0) = \text{sign}(x) \frac{(1 - 2\nu)(1 + \nu)}{E} \tau(x) - \frac{2(1 - \nu^2)}{\pi E} \int_{-b}^a \frac{p(s)}{x - s} ds$$

# Inverse problem

- Link displacement derivatives with tractions

$$\int_{-b}^a \frac{\tau(s)}{x-s} ds = -\frac{\pi(1-2\nu)}{2(1-\nu)} p(x) - \frac{\pi E}{2(1-\nu^2)} u_{x,x}(x,0)$$

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- If in contact interface we can prescribe  $p, u_{x,x}$  or  $\tau, u_{y,x}$ , then the problem reduces to

$$\int_{-b}^a \frac{\mathcal{F}(s)}{x-s} ds = \mathcal{U}(x)$$

- The general solution (case  $a = b$ ):

$$\mathcal{F}(x) = \frac{1}{\pi^2 \sqrt{a^2 - x^2}} \int_{-a}^a \frac{\sqrt{a^2 - s^2} \mathcal{U}(s) ds}{x-s} + \frac{C}{\pi \sqrt{a^2 - x^2}}, \quad C = \int_{-a}^a \mathcal{F}(s) ds$$

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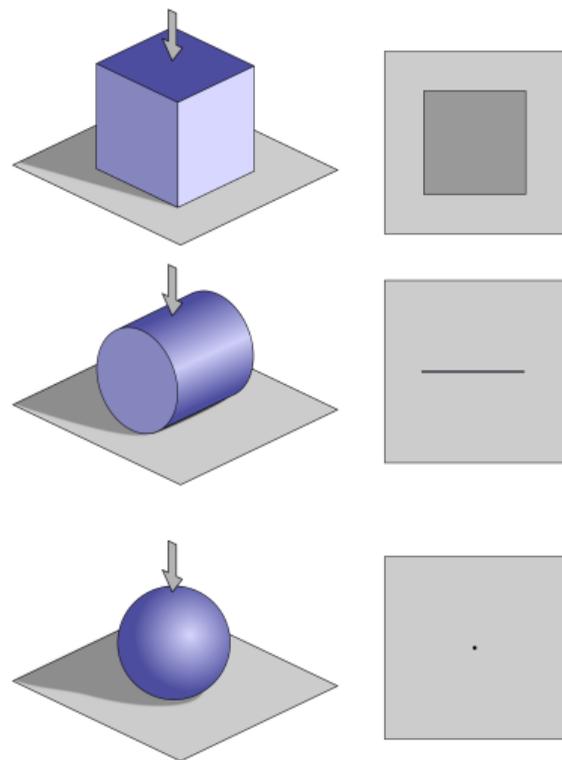
flat frictionless punch, consider P.V.

# Types of contact

- Known contact zone
  - conformal geometry  
*flat-to-flat, cylinder in a hole*
  - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone  
*general case*
- Point and line contact
- Frictionless  
*conservative, energy minimization problem*
- Frictional  
*path-dependent solution, from the first touch to the current moment*



Example

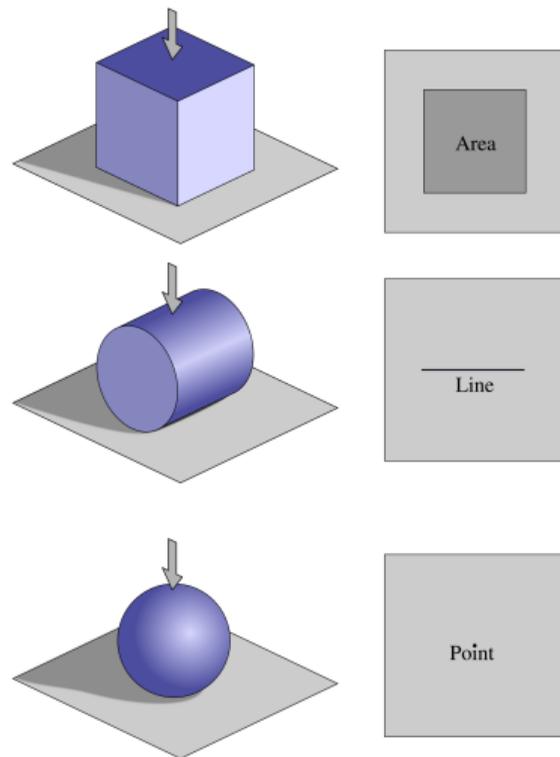


# Types of contact

- Known contact zone
  - conformal geometry  
*flat-to-flat, cylinder in a hole*
  - initially non-conformal geometry but huge pressure resulting in full contact
- Unknown contact zone  
*general case*
- Point and line contact
- Frictionless  
*conservative, energy minimization problem*
- Frictional  
*path-dependent solution, from the first touch to the current moment*



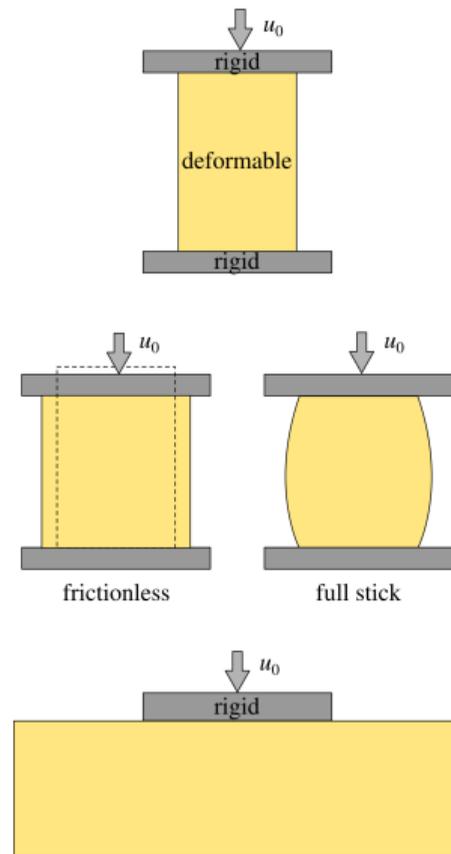
Example



# Analogy with boundary conditions

## Flat geometry

- Compression of a cylinder
- Frictionless  $u_z = u_0$
- Full stick conditions  $\underline{u} = u_0 \underline{e}_z$
- Rigid flat indenter  $u_z = u_0$



# Analogy with boundary conditions

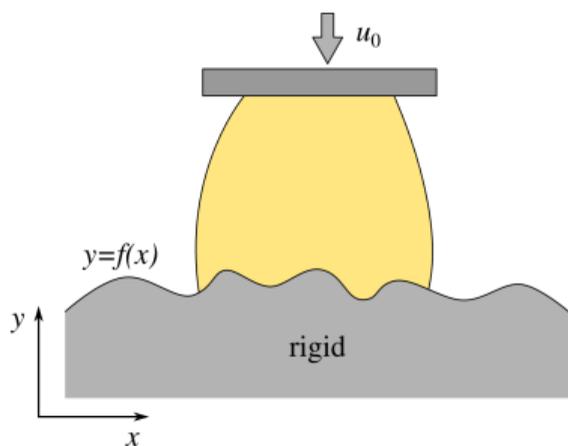
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- Rigid flat indenter  $u_z = u_0$

## Curved geometry

- Polar/spherical coordinates  $u_r = u_0$
- If frictionless contact on rigid surface  $y = f(x)$  is retained by high pressure

$$(\underline{X} + \underline{u}) \cdot \underline{e}_y = f((\underline{X} + \underline{u}) \cdot \underline{e}_x)$$



# Analogy with boundary conditions

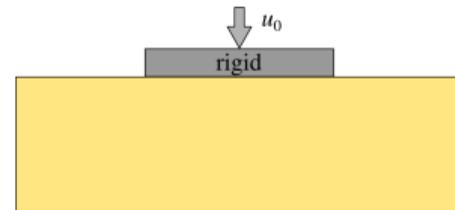
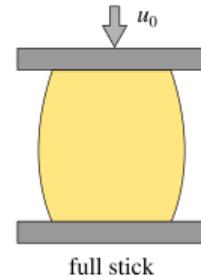
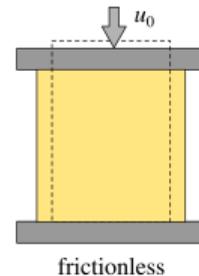
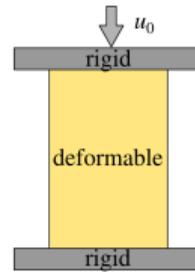
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# Analogy with boundary conditions II

## In general

- Type I: prescribed tractions

$$p(x, y), \tau_x(x, y), \tau_y(x, y)$$

- Type II: prescribed displacements

$$\underline{u}(x, y)$$

- Type III: tractions and displacements

$$u_z(x, y), \tau_x(x, y), \tau_y(x, y) \text{ or}$$

$$p(x, y), u_x(x, y), u_y(x, y)$$

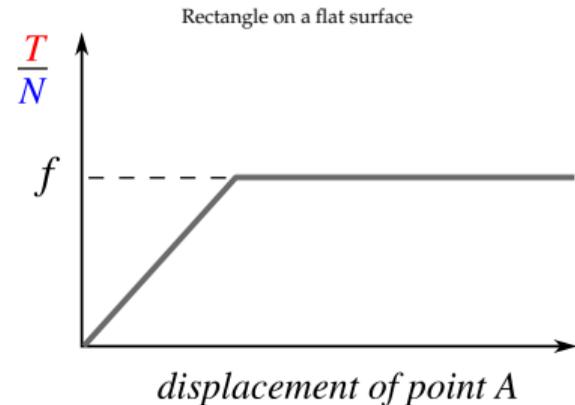
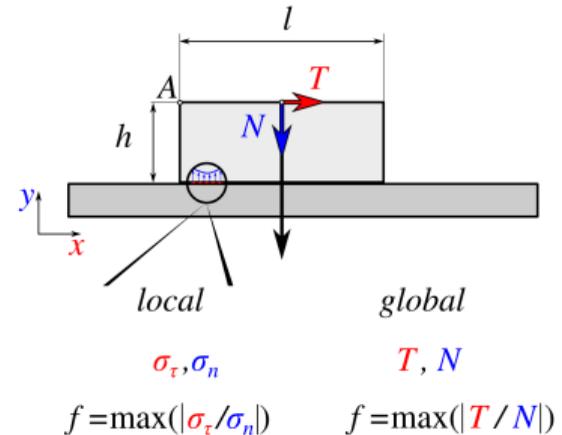
- Type IV: displacements and relation between tractions

$$u_z(x, y), \tau_x(x, y) = \pm f p(x, y)$$



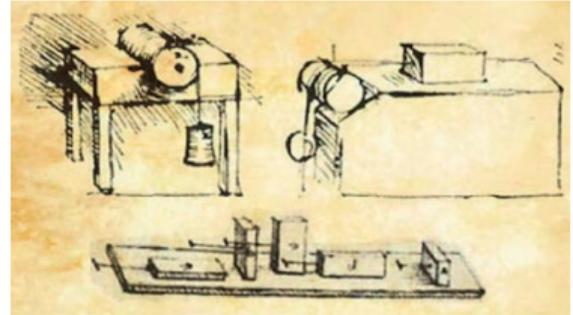
# Evidence of friction

- Existence of frictional resistance is evident
- Independence of the nominal contact area
- Globally:
  - stick:  $T < T_c(N)$
  - slip:  $T = T_c(N)$
- From experiments:
  - Threshold  $T_c \sim N$
  - Friction coefficient  $f = |T_c/N|$
- Locally
  - stick:  $\sigma_\tau < \tau_c(\sigma_n)$
  - slip:  $\sigma_\tau = f\sigma_n$



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First ever frictional experiments:  
notebook drawings of Leonardo da  
Vinci

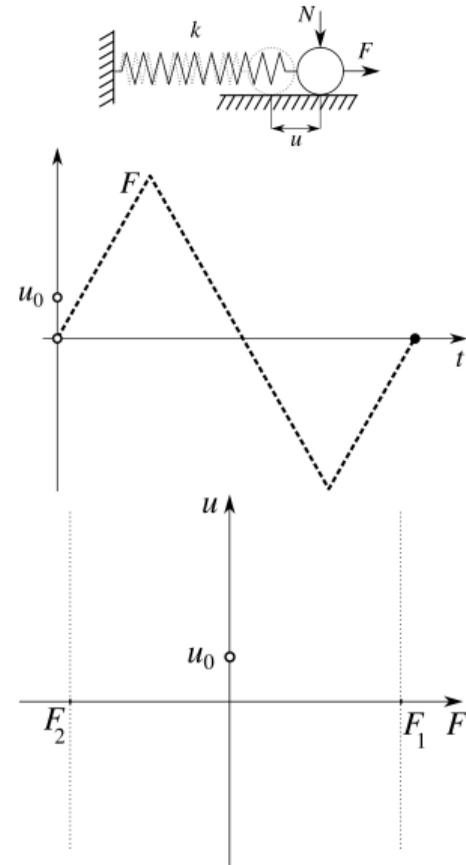
- proportionality between weight and frictional force
- friction is independent on the contact area

# Friction is not that intuitive

- Frictionless  
*conservative, energy minimization problem*
- Frictional  
*path-dependent solution, from the first touch to the current moment*



Example



# Friction is not that intuitive

What does your intuition says?

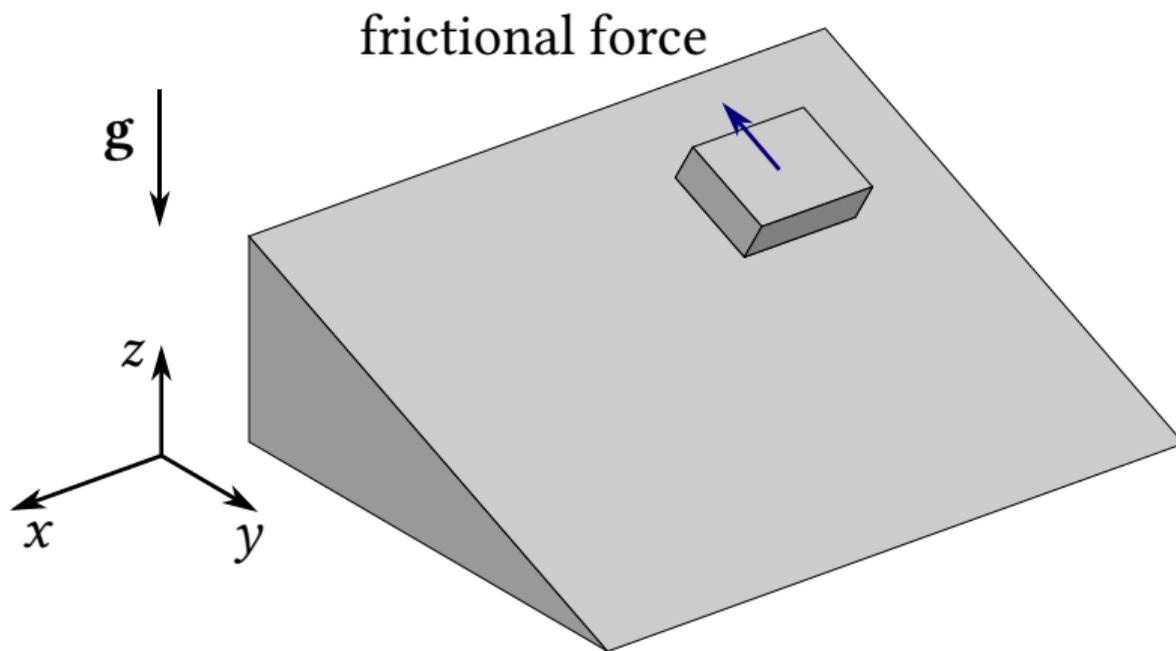


<https://forms.gle/cvnGs3r9JYRV9BWq9>

When the force starts to decrease, the contact point  
 goes to the left    does not move    goes to the right

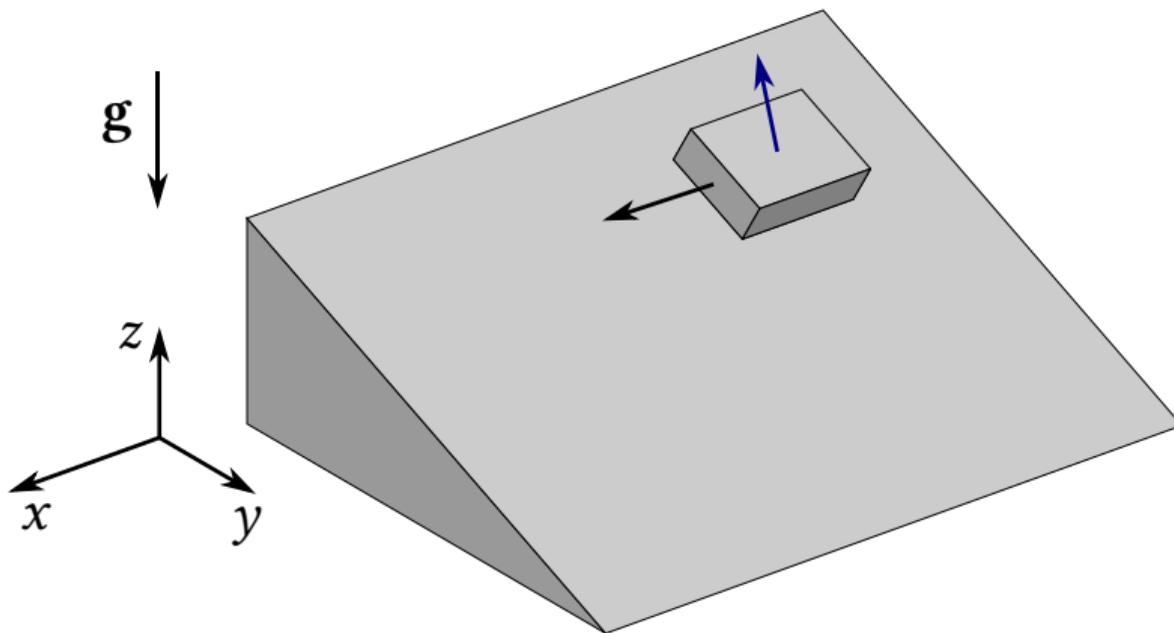
# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



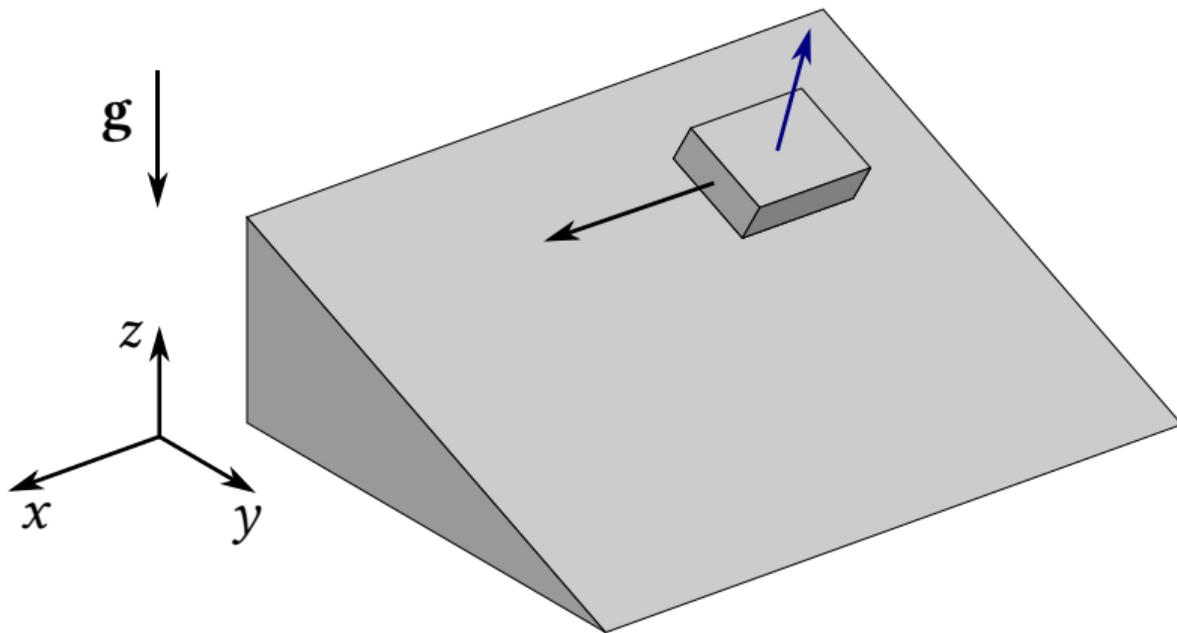
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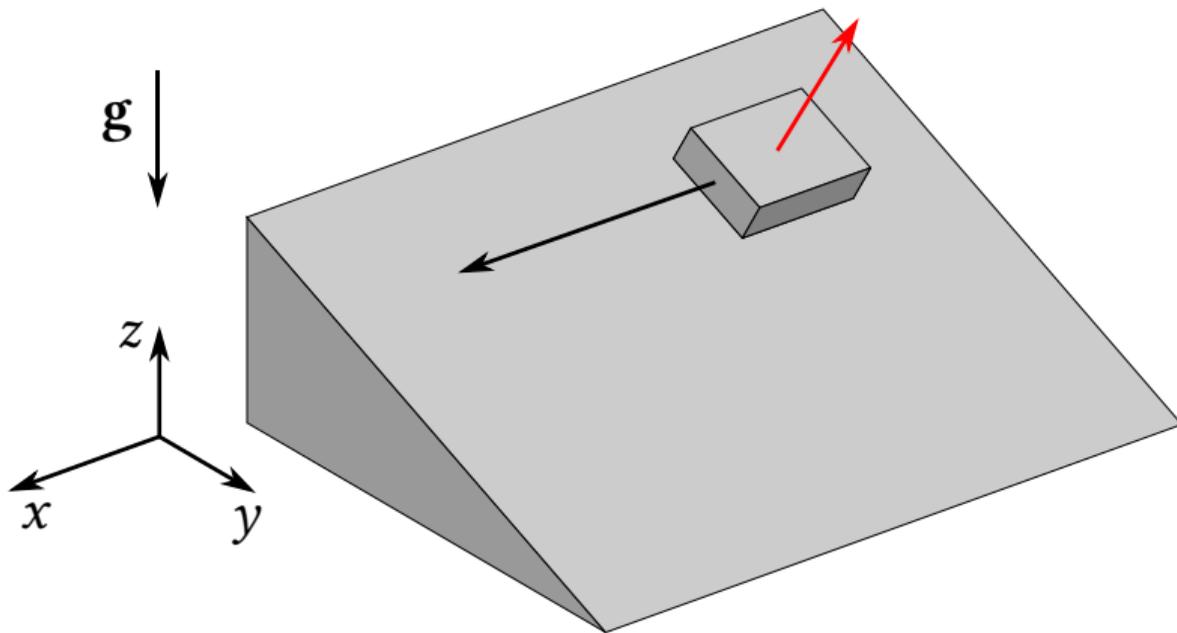
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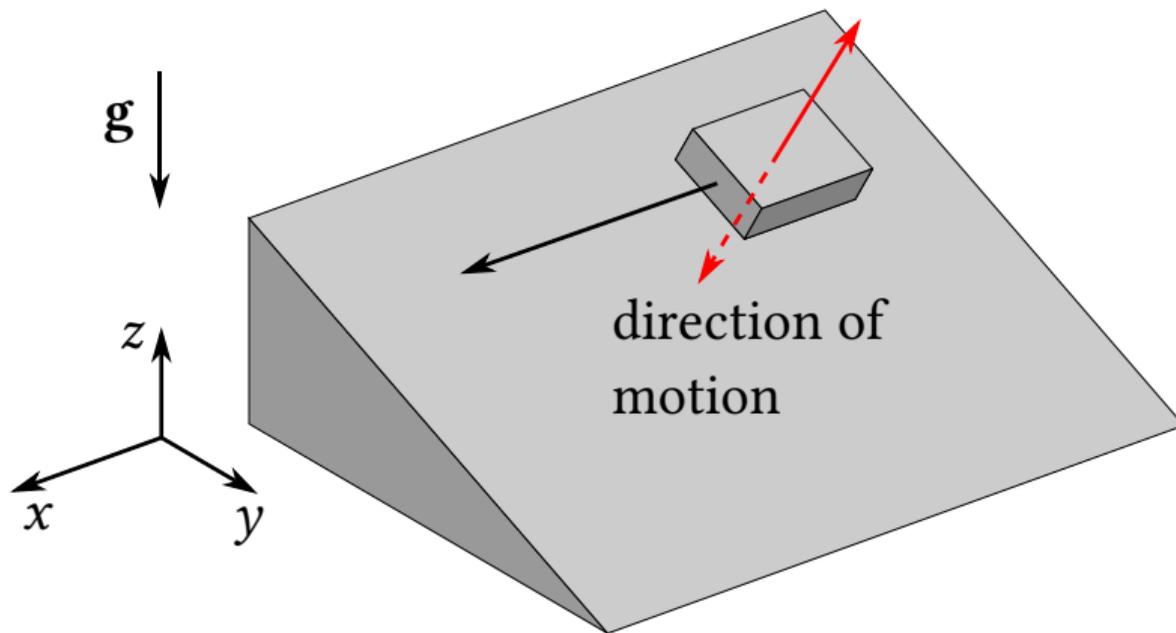
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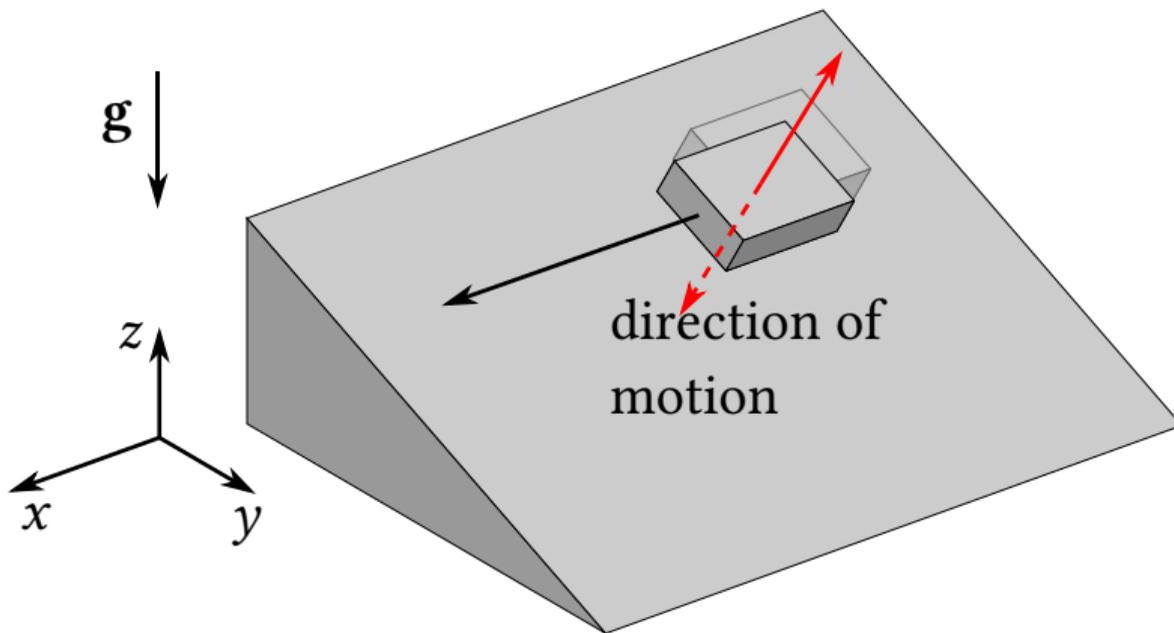
# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



# Direction of sliding

Sliding orthogonally to the slope of the inclined plane



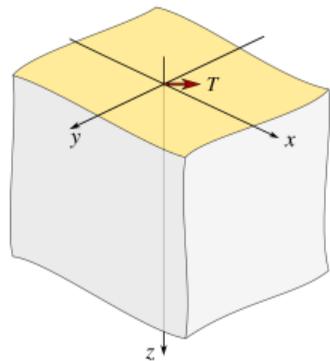
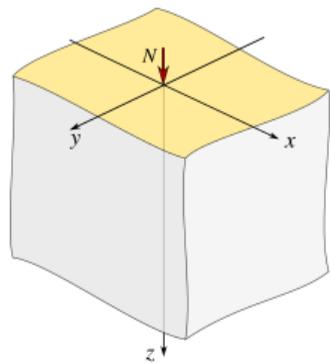
# Three-dimensional problem

- Analogy to Flamant's problem
- Potential functions of Boussinesq
- Boussinesq problem  
*concentrated normal force*
- Cerruti problem  
*concentrated tangential force*
- Displacements decay as  $\sim r^{-1}$

$$u_r(x, y, 0) = -\frac{1-2\nu}{4\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

$$u_z(x, y, 0) = \frac{1-\nu}{2\pi G} \frac{N}{\sqrt{x^2 + y^2}}$$

- Stress decay as  $\sim r^{-2}$
- Superposition principle



# Three-dimensional problem

## Normal force case

■ Full displacements:  $u_x = \frac{N}{4\pi G} \left( \frac{xz}{r^3} - (1-2\nu) \frac{x}{r(r+z)} \right)$

$$u_x = \frac{N}{4\pi G} \left( \frac{yz}{r^3} - (1-2\nu) \frac{y}{r(r+z)} \right)$$

$$u_z = \frac{N}{4\pi G} \left( \frac{z^2}{r^3} + \frac{2(1-\nu)}{r} \right)$$

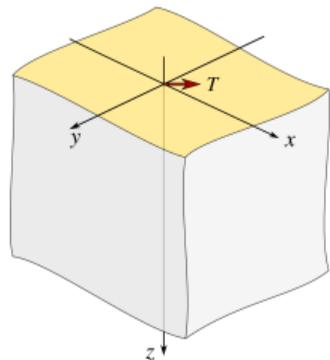
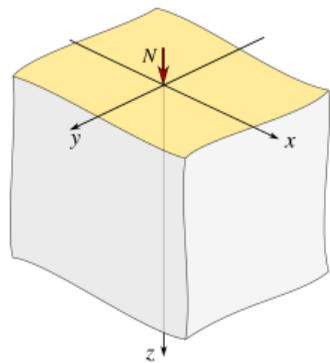
$$r = \sqrt{x^2 + y^2 + z^2}$$

■ Stresses:

$$\sigma_x = \frac{N}{2\pi} \left[ \frac{1-2\nu}{\rho^2} \left( \left(1 - \frac{z}{r}\right) \frac{x^2 - y^2}{\rho^2} + \frac{zy^2}{r^3} \right) - \frac{3zx^2}{r^5} \right]$$

$$\sigma_y = \frac{N}{2\pi} \left[ \frac{1-2\nu}{\rho^2} \left( \left(1 - \frac{z}{r}\right) \frac{y^2 - x^2}{\rho^2} + \frac{zx^2}{r^3} \right) - \frac{3zy^2}{r^5} \right] \quad \sigma_z = -\frac{3N}{2\pi} \frac{z^3}{r^5}$$

$$\rho = \sqrt{x^2 + y^2}$$



# Three-dimensional problem

## Normal force case

- Full displacements:  $u_x = \frac{N}{4\pi G} \left( \frac{xz}{r^3} - (1-2\nu) \frac{x}{r(r+z)} \right)$

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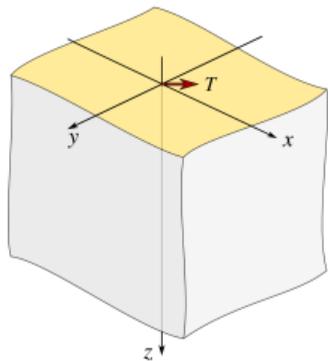
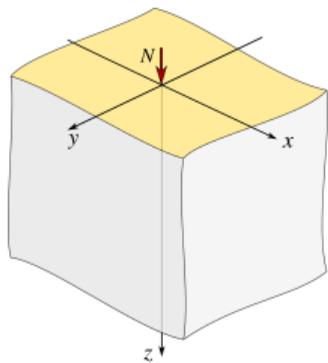
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$$\sigma_{xz} = -\frac{3N}{2\pi} \frac{xz^2}{r^5} \quad \sigma_{yz} = -\frac{3N}{2\pi} \frac{yz^2}{r^5}$$

$$\rho = \sqrt{x^2 + y^2}$$



# Hertzian contact

- No friction, no adhesion
- Two elastic materials  
 $E_1, \nu_1, E_2, \nu_2$
- Effective elastic modulus

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

- Two parabolic surfaces  
 $z_1 = ax_1^2 + by_1^2, z_2 = cx_2^2 + dy_2^2$
- Solids of revolution or cylinders  $R_1, R_2$ , effective curvature radius:

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Displacement and contact radius (only for 3D!):

$$\delta = a^2 / R^*$$

- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, \quad |r| < a$$

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## Ueber die Berührung fester elastischer Körper.

(Von Herrn Henrich Hertz.)

In der Theorie der Elasticität werden als Ursachen der Deformationen theils Kräfte, welche auf das Innere der Körper wirken, theils auf die Oberfläche wirkende Druckkräfte angenommen. Für beide Arten von Kräften kann der Fall eintreten, dass dieselben in einzelnen unendlich kleinen Theilen der Körper unendlich gross werden, so zwar, dass die Integrale der Kräfte über diese Theile genommen einen endlichen Werth behalten. Beschreiben wir alsdann um den Unstetigkeitspunkt eine geschlossene Fläche, deren Dimensionen sehr klein gegen die Dimensionen des ganzen Körpers sind, sehr gross hingegen im Vergleich zu den Dimensionen des Theils, in welchem die Kräfte angreifen, so können die Deformationen ausserhalb und innerhalb dieser Fläche ganz unabhängig von einander betrachtet werden. Ausserhalb hängen die Deformationen ab von der Gestalt des Gesamtkörpers, der Verteilung der übrigen Kräfte und den endlichen Integralen der Kraftcomponenten im Unstetigkeitspunkte, innerhalb hängen sie nur ab von der Verteilung der im Innern selbst angreifenden Kräfte. Die Drucke und Deformationen im Innern sind gegen die im Aeussern unendlich gross.

Im Folgenden wollen wir einen hieher gehörigen Fall behandeln, der praktisches Interesse hat\*, den Fall nämlich, dass zwei elastische isotrope Körper sich in einem sehr kleinen Theil ihrer Oberfläche berühren, und durch diesen Theil einen endlichen Druck der eine auf den andern ausüben. Die sich berührenden Oberflächen stellen wir uns als vollkommen glatt vor, d. h. wir nehmen nur einen senkrechten Druck zwischen den sich berührenden Theilen an. Das beiden Körpern nach der Deformation gemeinsame Stück der Oberfläche wollen wir die Druckfläche, die Begrenzung

\* Vgl. Weibull, Die Lehre von der Elasticität und Festigkeit, Prag 1887; I. p. 43. Grashof, Theorie der Elasticität und Festigkeit, Berlin 1878; p. 49–54.

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Original paper by Henrich Hertz "On the contact of elastic solids" (ENG trans.) (16 pages)

"His theory, worked out during the Christmas vacation 1880 at the age of 23(!), aroused considerable interest . . ." K.L. Johnson

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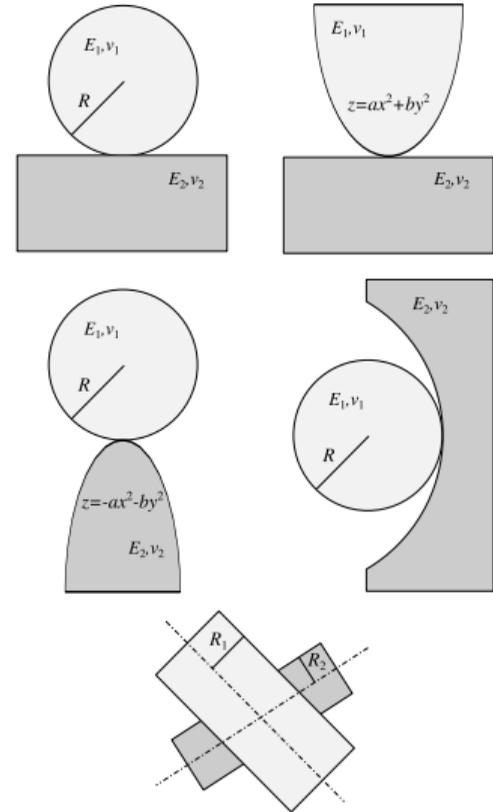
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$$\delta = a^2 / R^*$$

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$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, \quad |r| < a$$



Geometries resolved in the framework of Hertz theory

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- Contact pressure:

$$p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}, |r| < a$$

- Line contact (cylinders):

$$a = \left( \frac{4PR^*}{\pi E^*} \right)^{1/2}$$

$$p_0 = \frac{2P}{\pi a}$$

- Solids of revolution:

$$a = \left( \frac{3PR^*}{4E^*} \right)^{1/3}$$

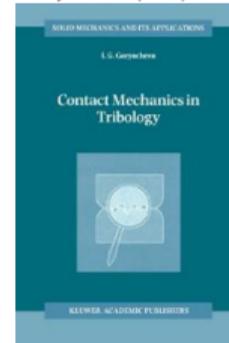
$$p_0 = \frac{3P}{2\pi a^2}$$

# Classical contact problems

- Various problems with rigid flat stamps:  
*circular, elliptic, frictionless, full-stick, finite friction*
- Hertz theory  
*normal frictionless contact of elastic solids*   
 $E_i, \nu_i$  and  $z_i = A_i x^2 + B_i y^2 + C_i xy, \quad i = 1, 2$
- Wedges (*coin*) and cones
- Circular inclusion in a conforming hole  
Steuermann, 1939, Goodman, Keer, 1965
- Frictional indentation  $z \sim x^n$   
Incremental approach Mossakovski, 1954  
self-similar solution Spence, 1968, 1975
- Adhesive contact Johnson et al, 1971, 1976
- Contact with layered materials (coatings)
- Elastic-plastic and viscoelastic materials
- Sliding/rolling of non-conforming bodies  
Cattaneo (1938), Mindlin (1949), Galin (1953), Goryacheva (1998)  
Note:  $u_r \sim (1 - 2\nu)/G$ , so if  $(1 - 2\nu_1)/G_1 = (1 - 2\nu_2)/G_2$  tangential tractions do not change normal ones



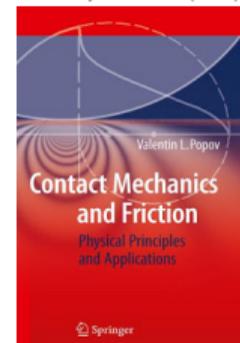
K.L. Johnson (1985)



I.G. Goryacheva (1998)



J.R. Barber (2018)



V.L. Popov (2017, 2nd Ed.)



Thank you for your attention!

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