

# Computational Contact Mechanics

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February 26, 2026



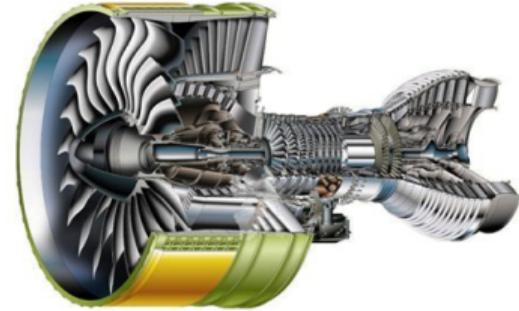
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V.A. Yastrebov, 2025

- Introduction
- Basics of Contact and Friction
- Towards a weak form
- Optimization methods
- Resolution algorithm
- Examples

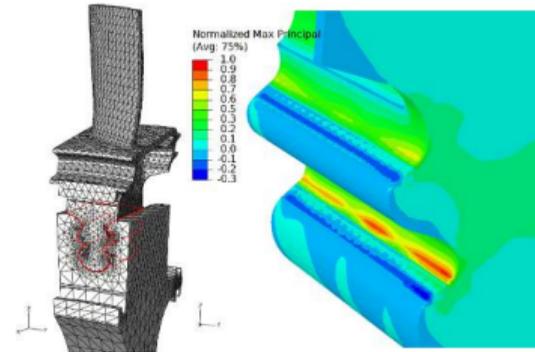
# Introduction

# Industrial and natural contact problems

## 1 Assembled parts, e.g. engines



Aircraft's engine GP 7200  
[www.safran-group.com](http://www.safran-group.com)



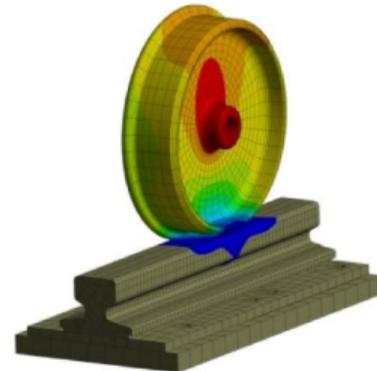
[1] M. W. R. Savage  
*J. Eng. Gas Turb. Power*, 134:012501 (2012)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts



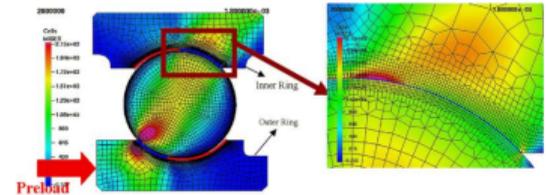
*High speed train TGV [www.sncf.com](http://www.sncf.com)*



*Wilde/ANSYS [wildeanalysis.co.uk](http://wildeanalysis.co.uk)*

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings



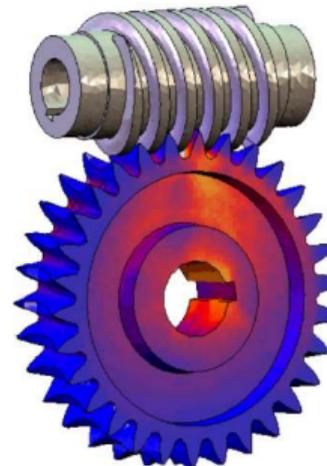
[1] F. Massi, J. Rocchi, A. Culla, Y. Berthier  
*Mech. Syst. Signal Pr.*, 24:1068-1080 (2010)

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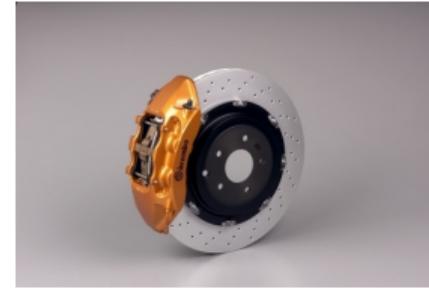
Helical gear [www.tpg.com.tw](http://www.tpg.com.tw)



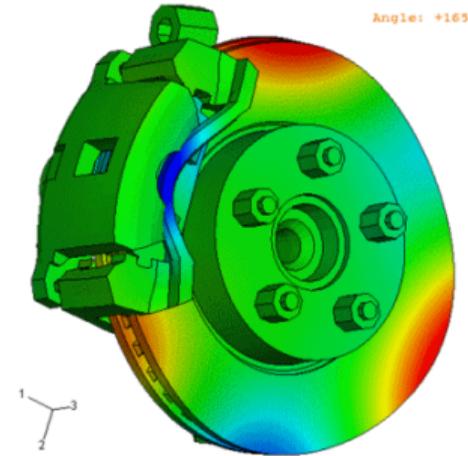
[www.mscsoftware.com](http://www.mscsoftware.com)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems



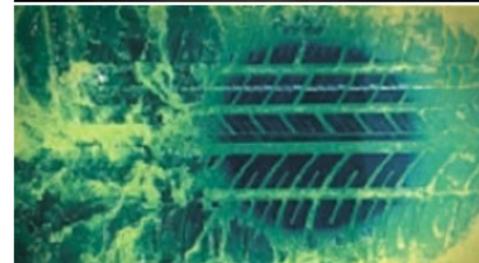
Assembled breaking system  
[www.brembo.com](http://www.brembo.com)



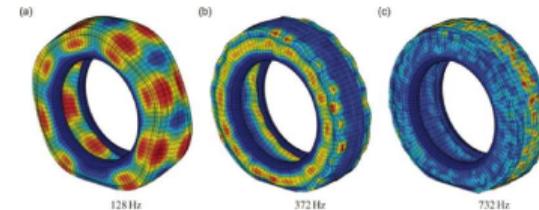
[www.mechanicalengineeringblog.com](http://www.mechanicalengineeringblog.com)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact



Tire-road contact [www.michelin.com](http://www.michelin.com)



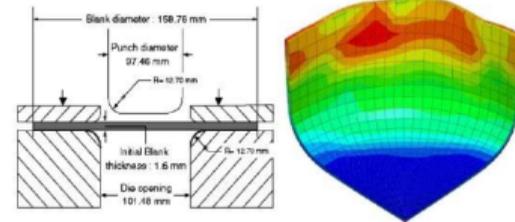
[1] M. Brinkmeier, U. Nackenhorst, S. Petersen, O. von Estorff, *J. Sound Vib.*, 309:20-39 (2008)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming



Deep drawing [www.thomasnet.com](http://www.thomasnet.com)



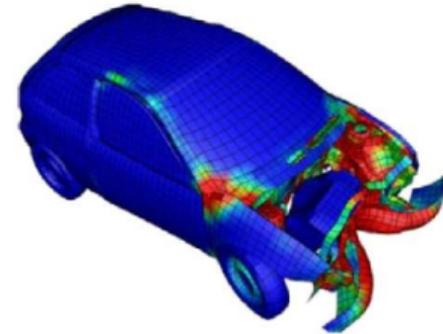
[1] G. Rousselier, F. Barlat, J. W. Yoon  
*Int. J. Plasticity*, 25:2383-2409 (2009)

# Industrial and natural contact problems

- 1 Assembled parts, e.g. engines
- 2 Railroad contacts
- 3 Gears and bearings
- 4 Breaking systems
- 5 Tire-road contact
- 6 Metal forming
- 7 Crash tests



*Crash-test [www.porsche.com](http://www.porsche.com)*



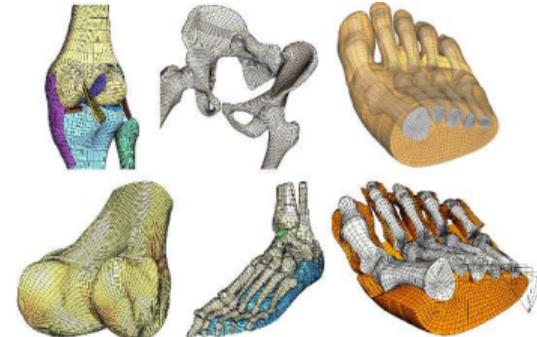
*[1] O. Klyavin, A. Michailov, A. Borovkov [www.fea.ru](http://www.fea.ru)*

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- 8 Biomechanics



Human articulations  
[www.sportssupplements.net](http://www.sportssupplements.net)



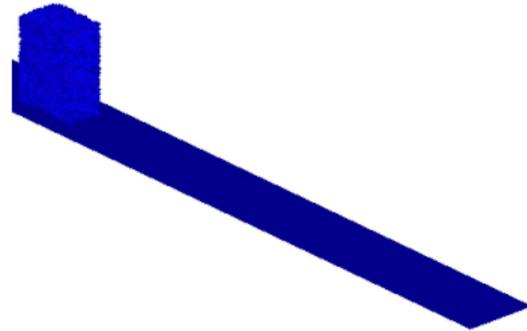
J. A. Weiss, University of Utah  
Musculoskeletal Research Laboratories

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- 7 Crash tests
- 8 Biomechanics
- 9 Granular materials



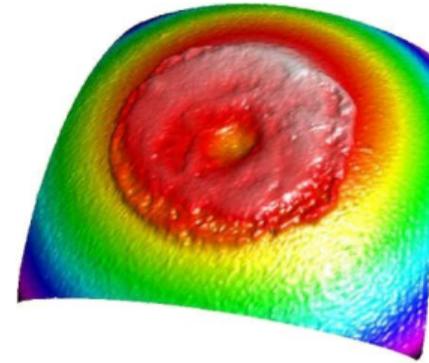
*Sand dunes [www.en.wikipedia.org](http://www.en.wikipedia.org)*



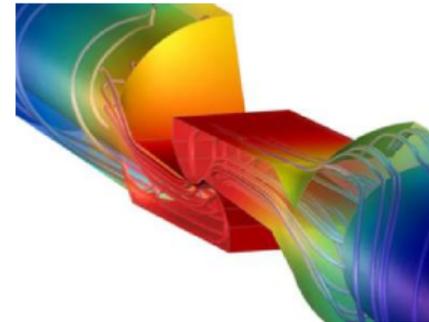
*E. Azema et al, LMGC90*

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- 7 Crash tests
- 8 Biomechanics
- 9 Granular materials
- 10 Electric contacts



*Damage at electric contact zone*  
[www.taicaan.com](http://www.taicaan.com)



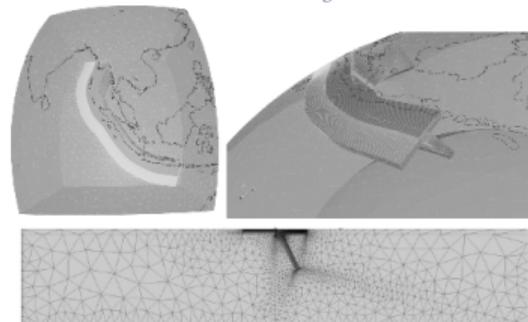
*Simulation of electric current*  
[www.comsol.com](http://www.comsol.com)

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- 11 Tectonic motions



*San-Andreas fault, by M. Rightmire  
[www.sciencedude.ocregister.com](http://www.sciencedude.ocregister.com)*



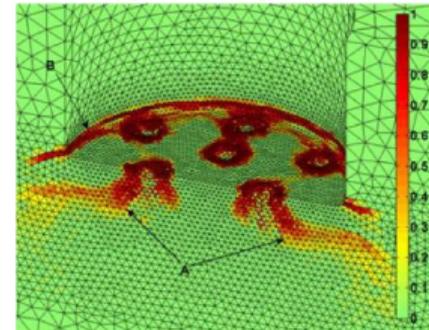
*[1] J.D. Garaud, L. Fleitout, G. Cailletaud  
Colloque CSMA (2009)*

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- 8 Biomechanics
- 9 Granular materials
- 10 Electric contacts
- 11 Tectonic motions
- 12 Deep drilling



Drill Bit tool *Robit Rocktools*;  
extraction of geothermal energy (*SINTEF, NTNU*)



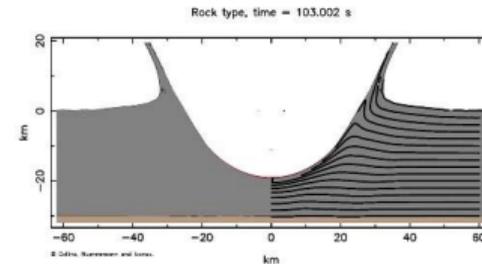
[1] T. Saksala, *Int. J. Numer. Anal. Meth. Geomech.* (2012)

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Impact crater, Arizona  
[www.MrEclipse.com](http://www.MrEclipse.com) et [maps.google.com](http://maps.google.com)



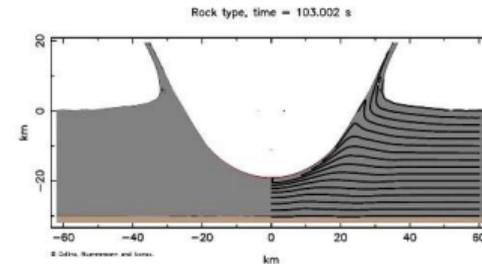
Simulation of formation of Copernicus crater  
Yue Z., Johnson B. C., et al. Projectile remnants in central peaks of lunar impact craters. *Nature Geo* 6 (2013)

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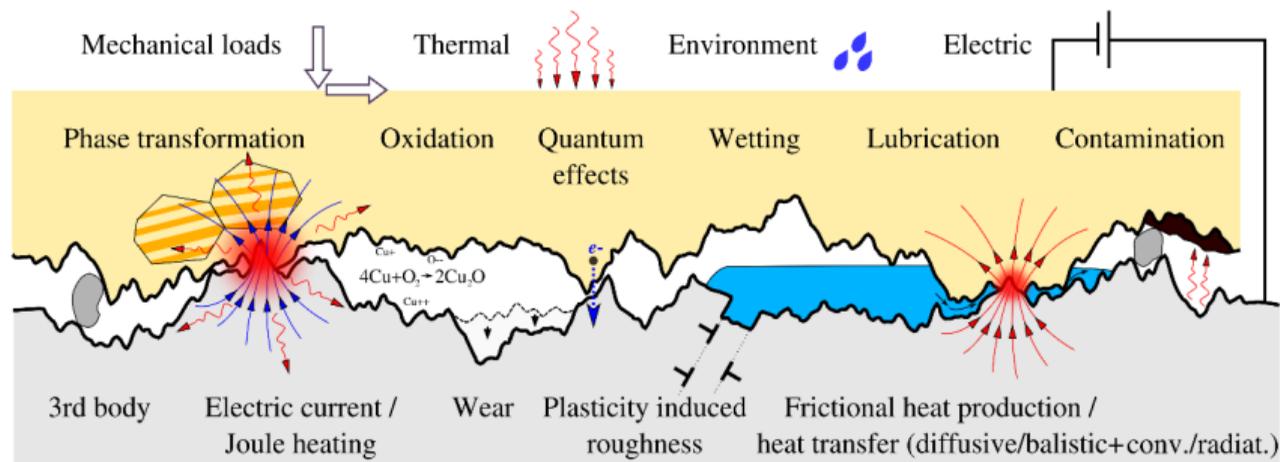
Impact crater, Arizona  
[www.MrEclipse.com](http://www.MrEclipse.com) et [maps.google.com](http://maps.google.com)



Simulation of formation of Copernicus crater  
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# Physical and mathematical complexity

- Contact interface is hard to observe in situ
- Plenty of phenomena happen in the interface
- Strong thermo-mechanical or fluid-solid coupling in sliding
- Mathematical formulation is also non-trivial, hard to handle analytically
- **Robust and accurate computational framework is needed**



Interface between two solids in contact

# Basics of Contact and Friction

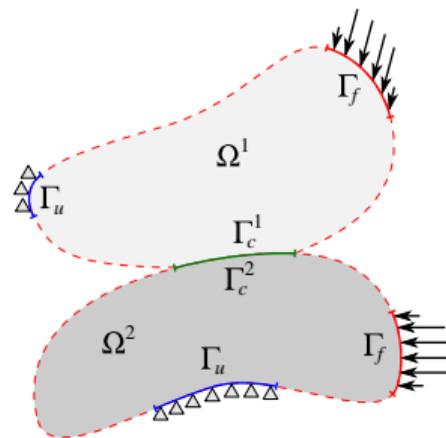
# Equilibrium and contact conditions

## ■ Balance of momentum

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 & \text{in } \Omega_{1,2} \\ \underline{\underline{\sigma}} \cdot \underline{\underline{n}} = \underline{\underline{t}}_0 & \text{on } \Gamma_f \\ \underline{\underline{u}} = \underline{\underline{u}}_0 & \text{on } \Gamma_u \\ \text{?} & \text{on } \Gamma_c \end{cases}$$

## ■ Frictionless contact conditions (*intuitive*)

- 1 No penetration
- 2 No adhesion
- 3 No shear transfer



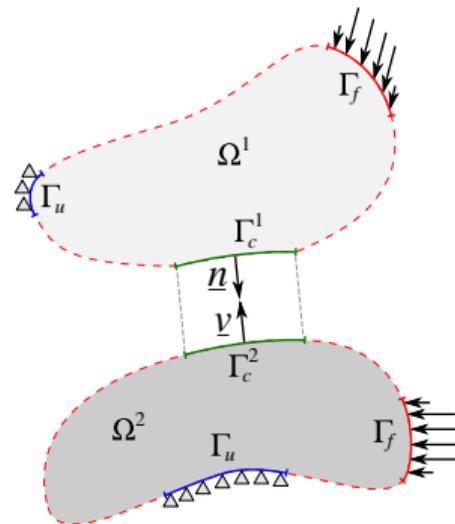
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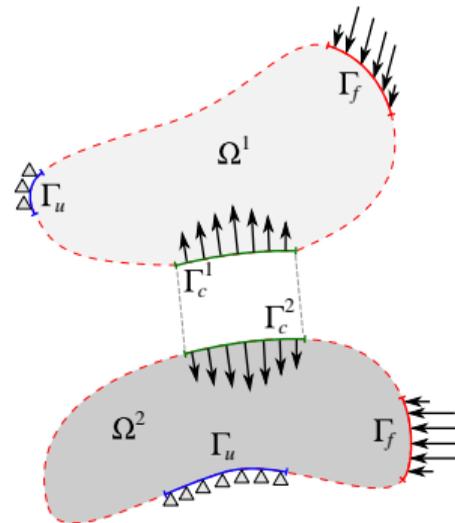
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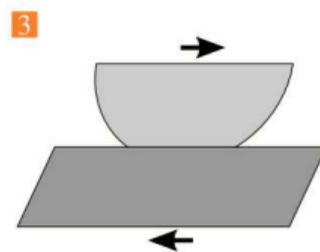
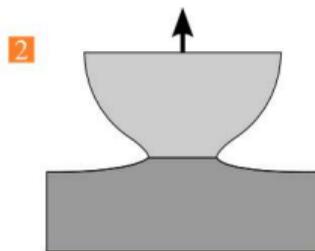
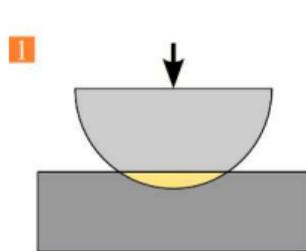
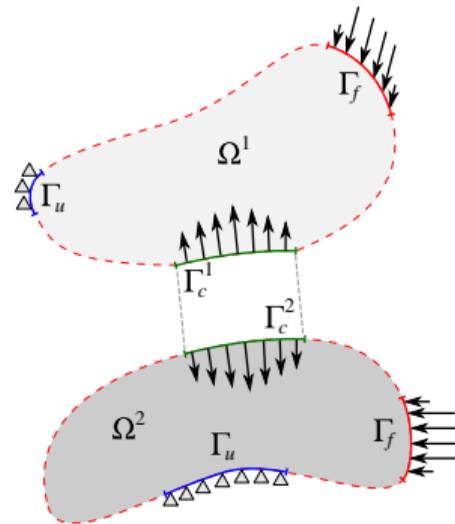
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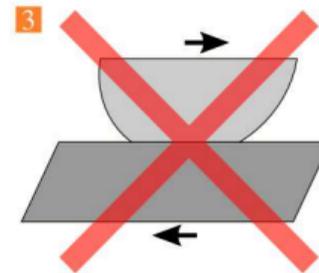
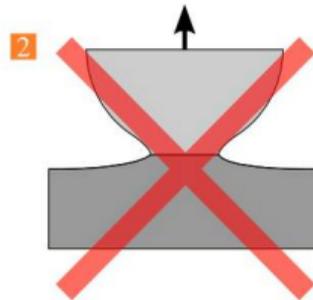
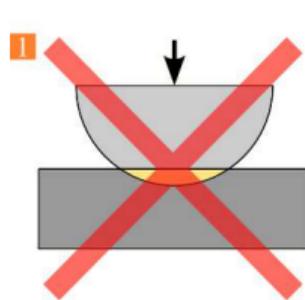
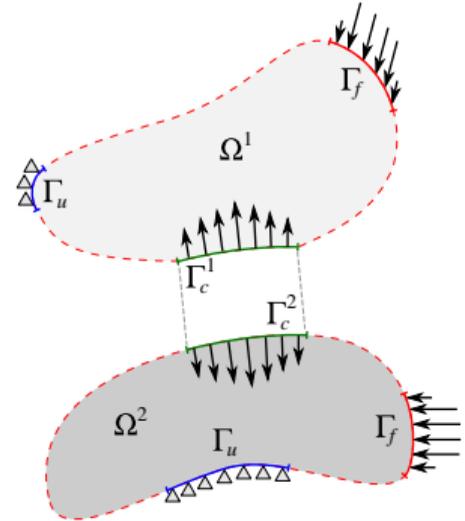
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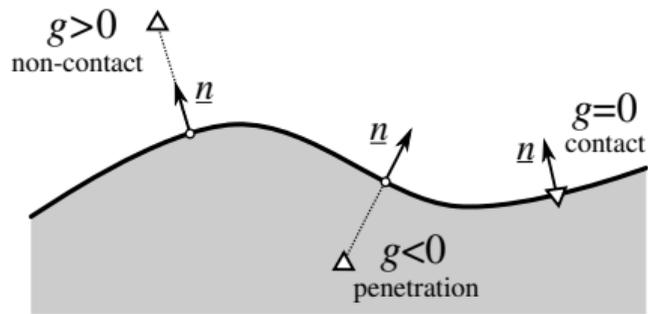
# Gap function

## ■ Gap function $g$

- gap = - penetration
- asymmetric function
- defined for
  - separation  $g > 0$
  - contact  $g = 0$
  - penetration  $g < 0$
- governs normal contact

## ■ Master and slave split

*Gap function is determined for all slave points with respect to the master surface*



*Gap between a slave point and a master surface*

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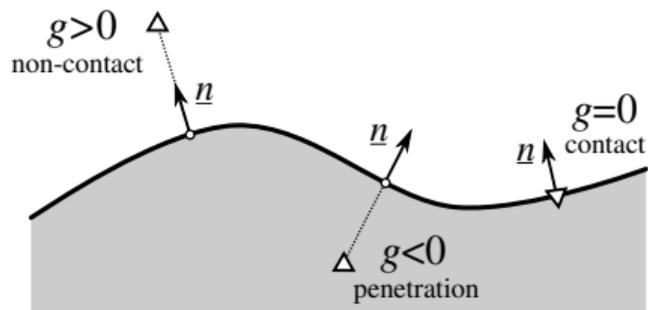
## ■ Master and slave split

*Gap function is determined for all slave points with respect to the master surface*

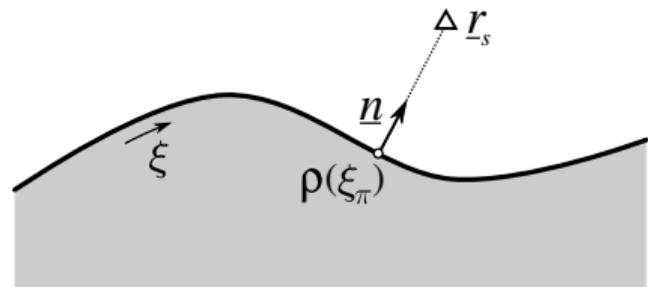
## ■ Normal gap

$$g_n = \underline{n} \cdot [\underline{r}_s - \underline{\rho}(\xi_\pi)],$$

$\underline{n}$  is a unit normal vector,  $\underline{r}_s$  slave point,  $\underline{\rho}(\xi_\pi)$  projection point on the master surface



*Gap between a slave point and a master surface*



*Definition of the normal gap*

Consider existence and uniqueness



# Frictionless or normal contact conditions

- **No penetration**

*Always non-negative gap*

$$g \geq 0$$

- **No adhesion**

*Always non-positive contact pressure*

$$\sigma_n^* \leq 0$$

- **Complementarity condition**

*Either zero gap and non-zero pressure, or non-zero gap and zero pressure*

$$g \sigma_n = 0$$

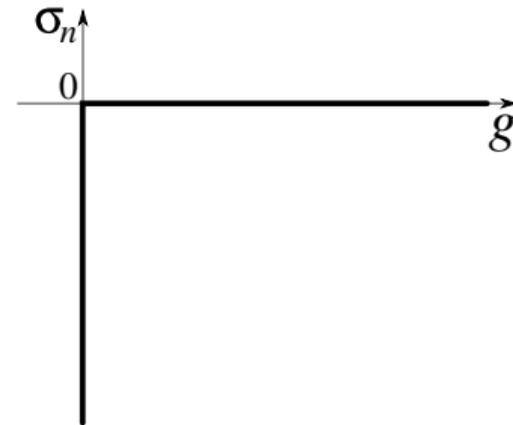
- **No shear transfer (automatically)**

$$\underline{\sigma}_t^{**} = 0$$

---

$$\sigma_n^* = (\underline{\sigma} \cdot \underline{n}) \cdot \underline{n} = \underline{\sigma} : (\underline{n} \otimes \underline{n})$$

$$\underline{\sigma}_t^{**} = \underline{\sigma} \cdot \underline{n} - \sigma_n \underline{n} = \underline{n} \cdot \underline{\sigma} \cdot (\underline{I} - \underline{n} \otimes \underline{n})$$



Scheme explaining normal contact conditions

# Frictionless or normal contact conditions

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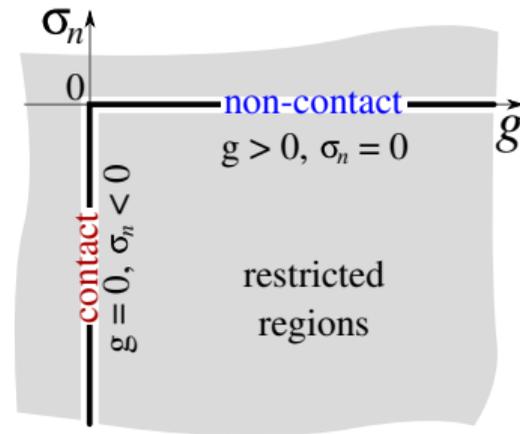
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**Improved** scheme explaining normal contact conditions

# Frictionless or normal contact conditions

In mechanics:

*Normal contact conditions*

≡

*Frictionless contact conditions*

≡

*Hertz<sup>[1]</sup>-Signorini<sup>[2]</sup> conditions*

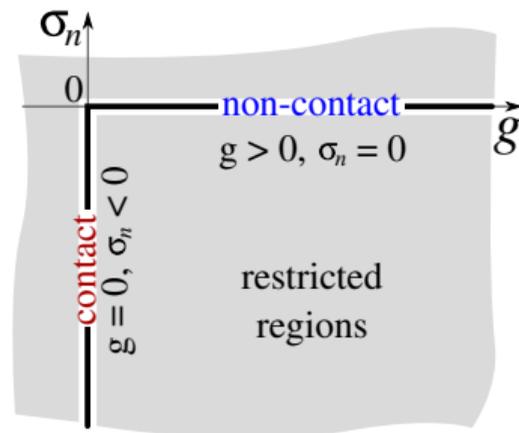
≡

*Hertz<sup>[1]</sup>-Signorini<sup>[2]</sup>-Moreau<sup>[3]</sup> conditions*

also known in **optimization theory** as

*Karush<sup>[4]</sup>-Kuhn<sup>[5]</sup>-Tucker<sup>[6]</sup> (KKT) conditions*

$$g \geq 0, \quad \sigma_n \leq 0, \quad g\sigma_n = 0$$



**Improved** scheme explaining normal contact conditions

<sup>1</sup>Heinrich Rudolf Hertz (1857–1894) a German physicist who first formulated and solved the frictionless contact problem between elastic ellipsoidal bodies.

<sup>2</sup>Antonio Signorini (1888–1963) an Italian mathematical physicist who gave a general and rigorous mathematical formulation of contact constraints.

<sup>3</sup>Jean Jacques Moreau (1923–2014) a French mathematician who formulated a non-convex optimization problem based on these conditions and introduced pseudo-potentials in contact mechanics.

<sup>4</sup>William Karush (1917–1997), <sup>5</sup>Harold William Kuhn (1925–2014) American mathematicians,

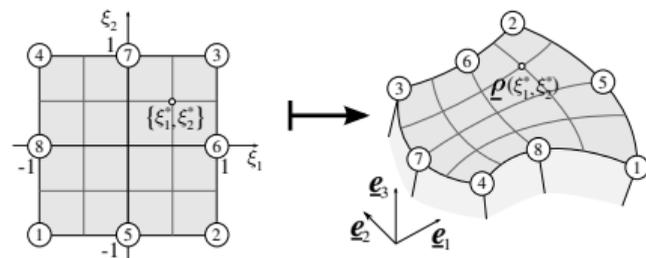
<sup>6</sup>Albert William Tucker (1905–1995) a Canadian mathematician.

# Relative sliding

## Recall:

- Convective coordinate in parent space  $\xi_i \in (-1; 1)$
- Mapping to real space

$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$



# Relative sliding

## Recall:

- Convective coordinate in parent space  $\xi_i \in (-1; 1)$
- Mapping to real space

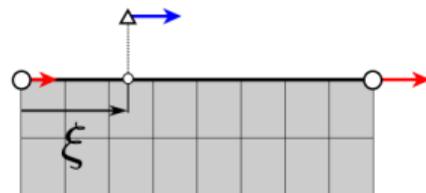
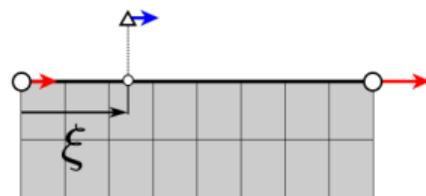
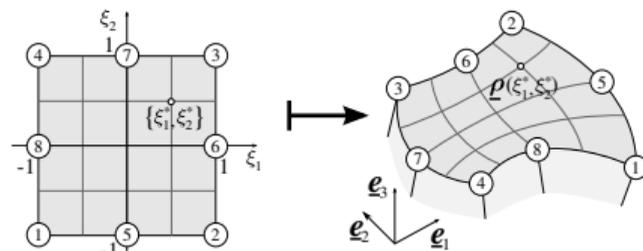
$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$

### ■ Tangential slip velocity $\underline{v}_t$ must take into account:

- only tangential component
- relative rigid body motion
- master's deformation

$$\underline{v}_t = \frac{\partial \underline{\rho}}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial \underline{\rho}}{\partial \xi_2} \dot{\xi}_2$$

where  $\partial \underline{\rho} / \partial \xi_i$  are the tangent vectors of the local basis and  $\xi_i$  are the convective coordinates.



*Relative slip between a slave point and a deformable master surface*

# Relative sliding

## Recall:

- Convective coordinate in parent space  $\xi_i \in (-1; 1)$
- Mapping to real space

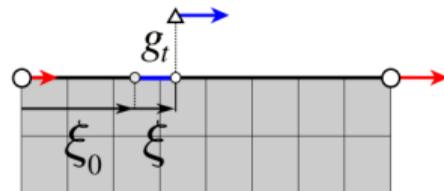
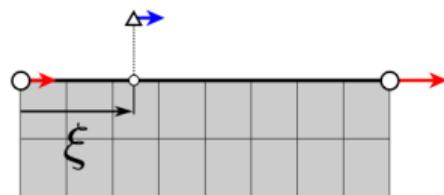
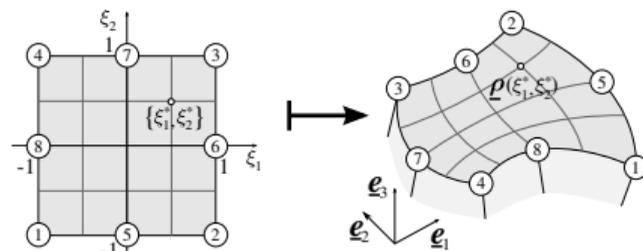
$$\underline{\rho}(\xi_1, \xi_2, t) = \sum_{i=1}^8 N^i(\xi_1, \xi_2) \underline{\rho}^i(t)$$

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- only tangential component
- relative rigid body motion
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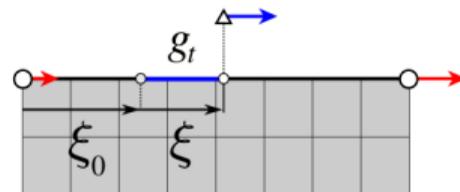
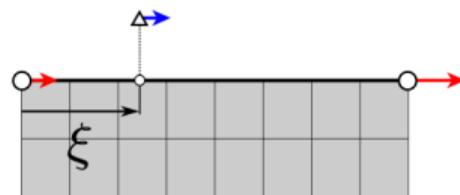
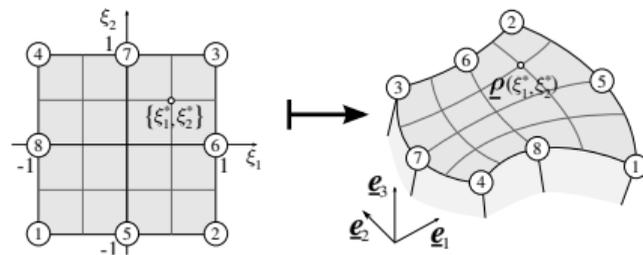
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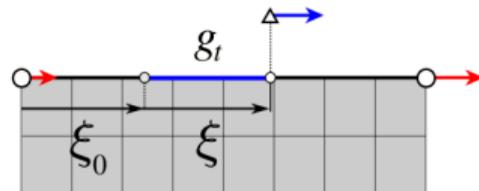
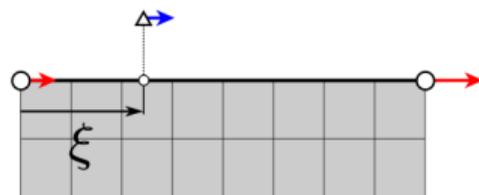
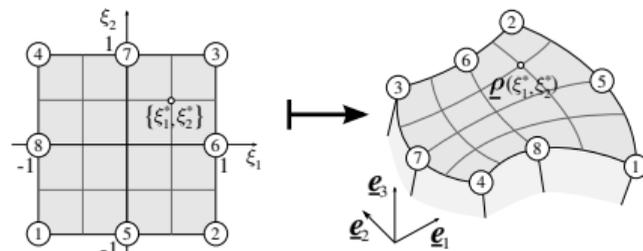
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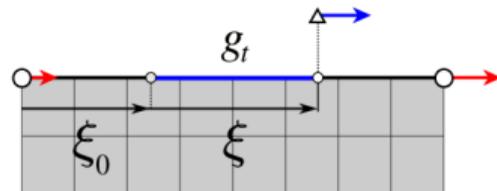
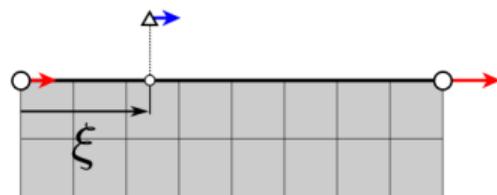
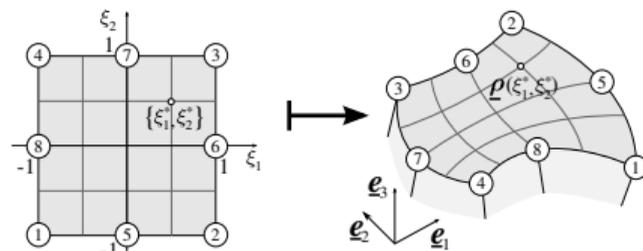
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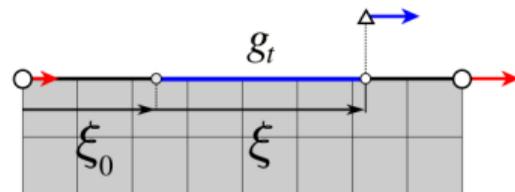
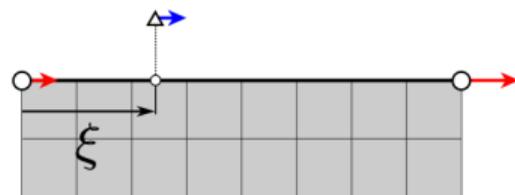
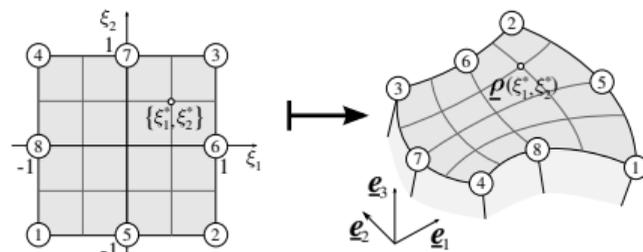
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Relative slip between a slave point and a deformable master surface

# Relative sliding: example

Consider a one-dimensional example:

$P$  is a projection of  $A$  on segment  $BC$ .

$$x_P = \xi x_C + (1 - \xi)x_B \quad (1)$$

Velocity of the projection point

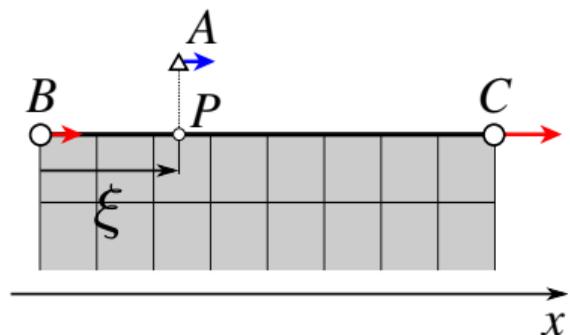
$$\dot{x}_P = \underbrace{\xi \dot{x}_C + (1 - \xi) \dot{x}_B}_{\frac{\partial x_P}{\partial t}} + \underbrace{(x_C - x_B) \dot{\xi}}_{\frac{\partial x_P}{\partial \xi} \dot{\xi}}$$

Subtract the velocity of point  $x_P$  for fixed  $\xi$

$$v_t = \dot{x}_P - \frac{\partial x_P}{\partial t} = (x_C - x_B) \dot{\xi} = \frac{\partial x}{\partial \xi} \dot{\xi}$$

Compute tangential slip increment

$$\Delta g_t^{n+1} = \left. \frac{\partial x}{\partial \xi} \right|_{\xi^n} (\xi^{n+1} - \xi^n)$$



Example of a one-dimensional relative slip

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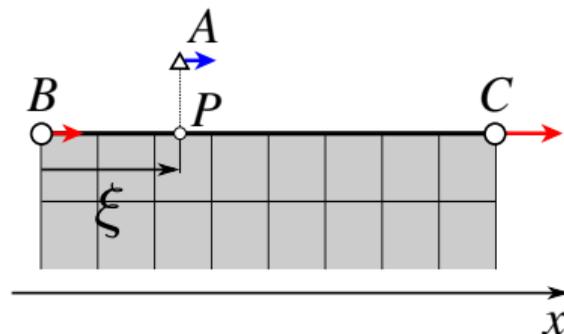
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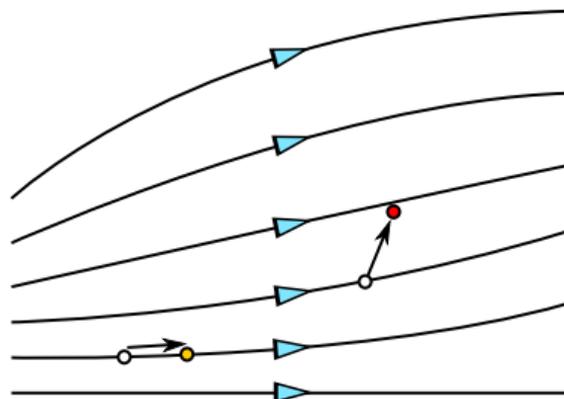
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Example of a one-dimensional relative slip



Fisherman's analogy: observing sea flow around the boat.  
Lie derivative: the change of a vector field along the change of another vector field

# Amontons-Coulomb's friction

- **No contact**  $g > 0, \sigma_n = 0$
- **Stick**  $|\underline{v}_t| = 0$   
*Inside slip surface/Coulomb's cone*

$$f = |\underline{\sigma}_t| - \mu|\sigma_n| < 0$$

- **Slip**  $|\underline{v}_t| > 0$   
*On slip surface/Coulomb's cone*

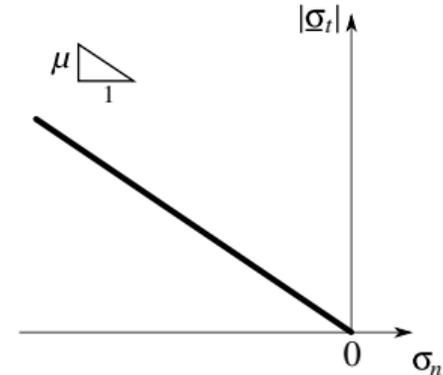
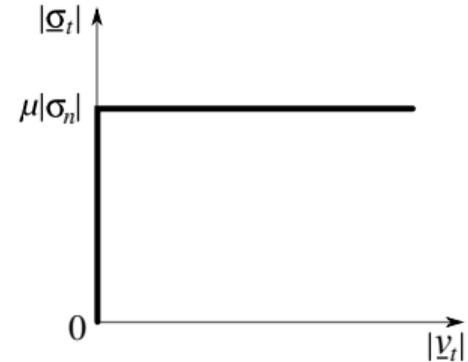
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- **Complementary condition**  
*One is zero another one is not or vice versa*

$$|\underline{v}_t| \left( |\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0$$

- **Direction of friction**  
*Shear and sliding are collinear*

$$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme explaining frictional contact conditions

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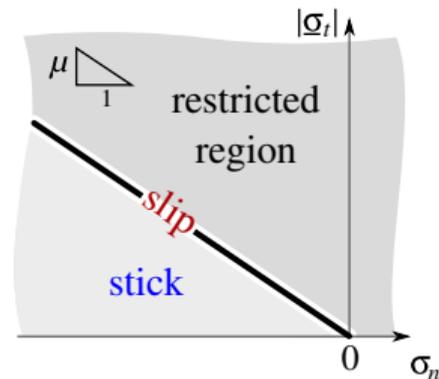
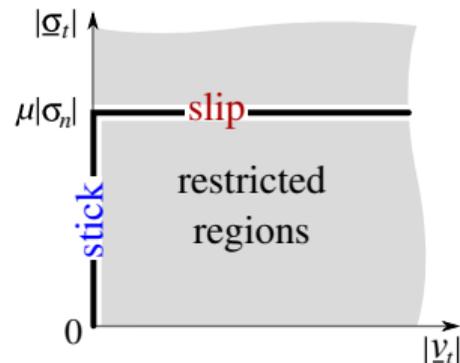
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**Improved** scheme explaining frictional contact conditions

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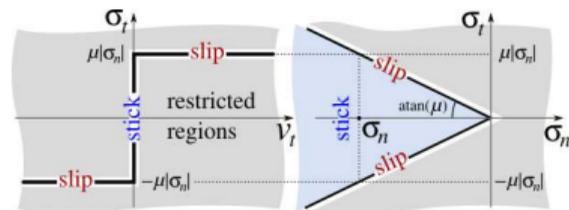
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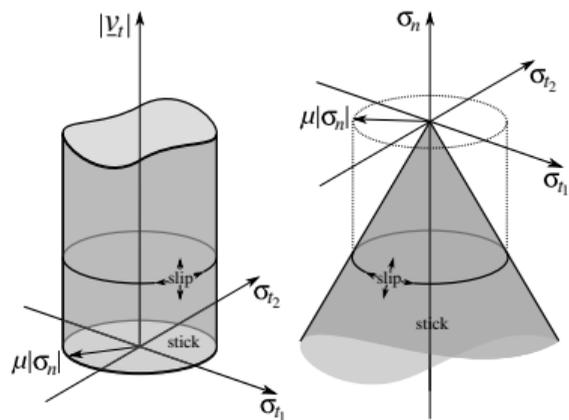
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$$\underline{v}_t \parallel \underline{\sigma}_t$$



Scheme of 2D frictional contact

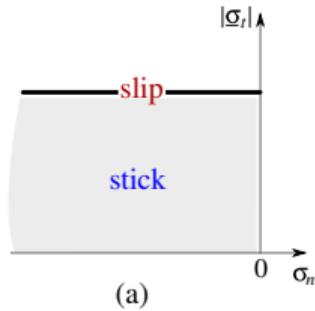


Scheme of 3D frictional contact

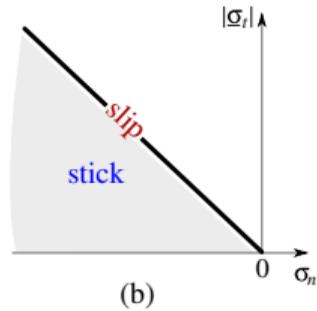
$$|\underline{v}_t| \geq 0, \quad |\underline{\sigma}_t| - \mu|\sigma_n| \leq 0, \quad |\underline{v}_t| \left( |\underline{\sigma}_t| - \mu|\sigma_n| \right) = 0, \quad \frac{\underline{\sigma}_t}{|\underline{\sigma}_t|} = - \frac{\underline{v}_t}{|\underline{v}_t|}$$

# More friction laws

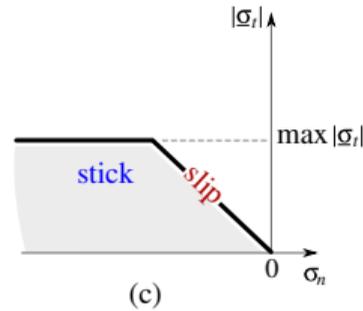
## • Static criteria



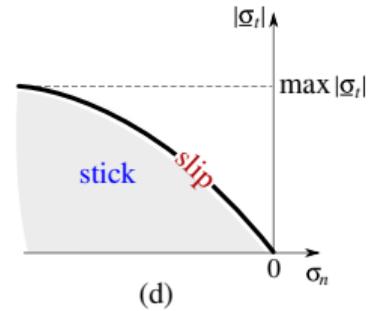
(a) Tresca



(b) Amontons-Coulomb

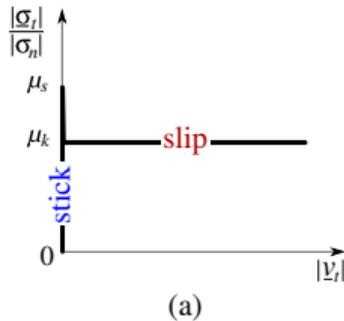


(c) Coulomb-Orowan

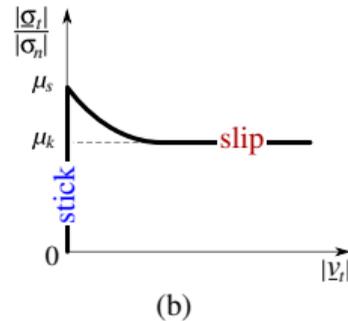


(d) Shaw

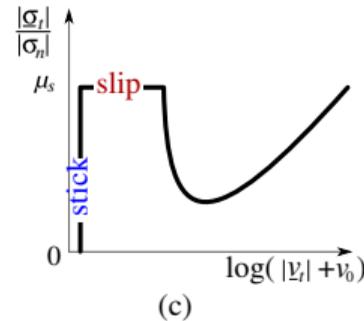
## • Kinetic criteria



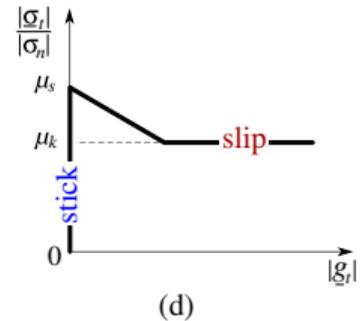
(a,b) velocity weakening



(c) velocity weakening-strengthening



(c)



(d) linear slip weakening

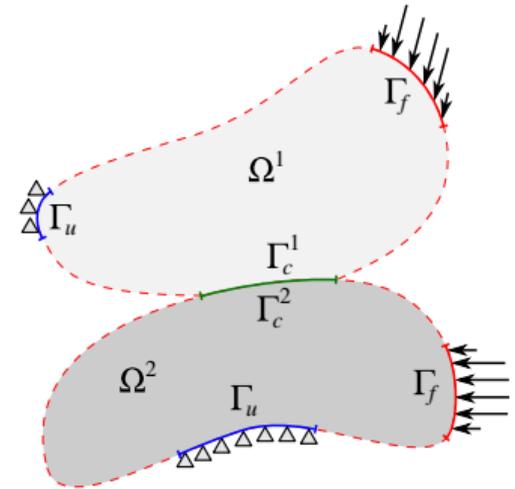
- $\mu_s$  static and  $\mu_k$  kinetic coefficients of friction.

# Towards a weak form

# From strong to a weak form

- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$



*Two solids in contact*

# From strong to a weak form

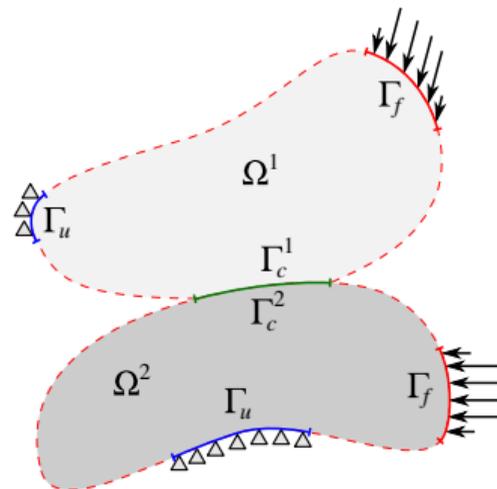
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- Balance of virtual works



$$\boxed{\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma} + \int_{\Omega} [\underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} - \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}}] d\Omega = 0$$



*Two solids in contact*

# From strong to a weak form

- Balance of momentum and boundary conditions

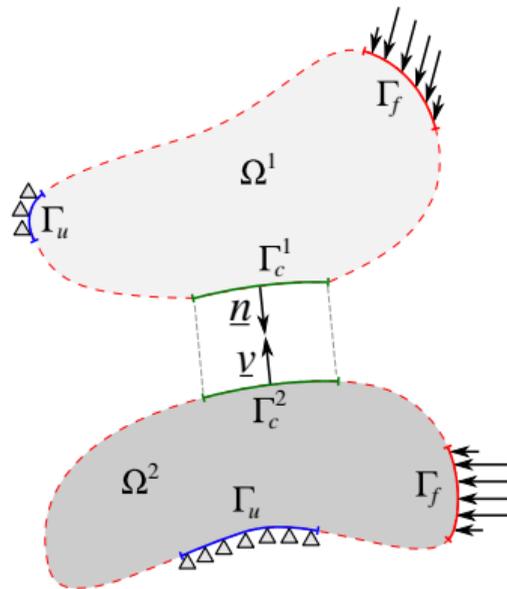
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$$\int_{\partial\Omega} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{u}} d\Gamma =$$

$$\int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{\rho}} d\bar{\Gamma}_c^1 + \int_{\bar{\Gamma}_c^2} \underline{\underline{v}} \cdot \underline{\underline{\sigma}} \cdot \delta \underline{\underline{r}} d\bar{\Gamma}_c^2 + \int_{\bar{\Gamma}_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma_f$$



Two solids in contact

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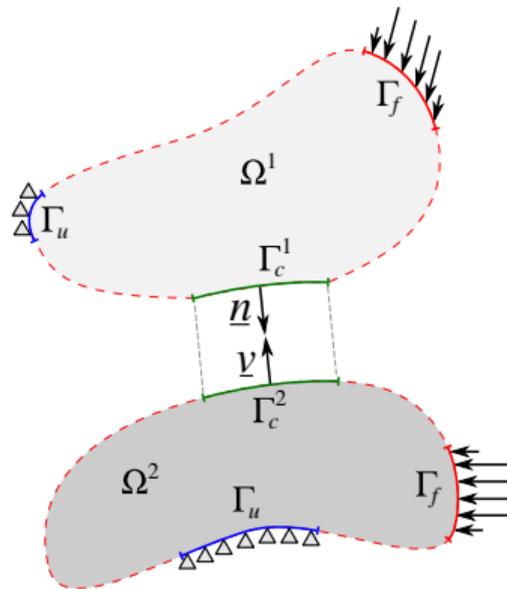
- Balance of virtual works



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$$= \int_{\bar{\Gamma}_c^1} \underline{\underline{n}} \cdot \underline{\underline{\sigma}} \cdot \delta(\underline{\underline{\rho}} - \underline{\underline{r}}) d\bar{\Gamma}_c^1 = \int_{\bar{\Gamma}_c^1} (\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}}) d\bar{\Gamma}_c^1$$



Two solids in contact

# From strong to a weak form

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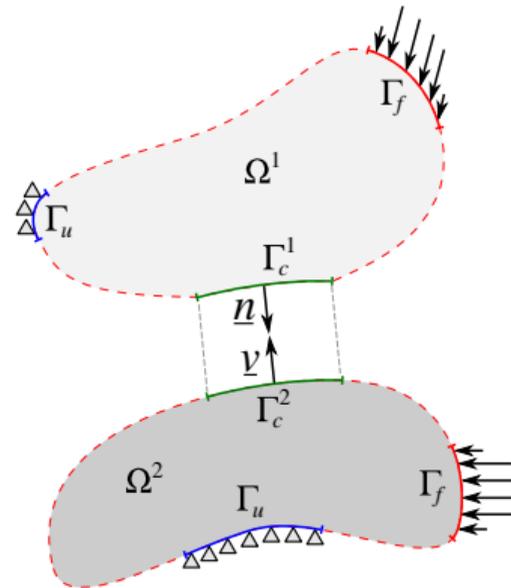
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$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \underbrace{\int_{\bar{\Gamma}_c^1} (\sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}}) d\bar{\Gamma}_c^1}_{\text{Contact term}} = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega$$

Contact term



Two solids in contact



# From strong to a weak form

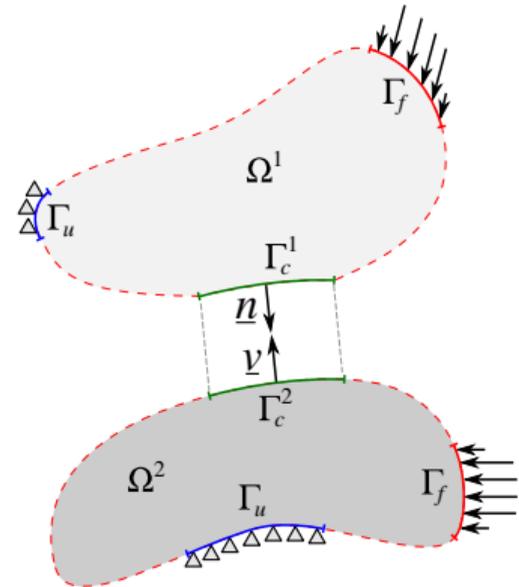
- Balance of momentum and boundary conditions

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{f}}_v = 0 \quad \text{in } \Omega = \Omega_1 \cup \Omega_2 \quad + \text{B.C.}$$

- Balance of virtual works

$$\underbrace{\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega}_{\text{Change of the internal energy}} + \underbrace{\int_{\tilde{\Gamma}_c^1} \left( \sigma_n \delta g_n + \underline{\underline{\sigma}}_t^T \delta \underline{\underline{\xi}} \right) d\bar{\Gamma}_c^1}_{\text{Contact term}} \geq$$

$$\underbrace{\int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma}_{\text{Virtual work of external forces}} + \underbrace{\int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega}_{\text{Virtual work of volume forces}}$$



Two solids in contact

- Functional **subspace**

$\delta \underline{\underline{u}}, \underline{\underline{u}} \in \mathbb{H}^1(\Omega)$  Sobolev space of the first order (function and its first derivate is square integrable) and  $\underline{\underline{u}}$  satisfy boundary conditions and **contact conditions**, so we do optimization on a subset of  $\mathbb{H}^1(\Omega)$ .

# Variational inequality

- Optimization problem for  $F : \mathbb{V} \rightarrow \mathbb{R}$
- Find  $u \in \mathbb{V}$  s.t.  $\forall v \in \mathbb{V} : F(u) \leq F(v)$
- If  $F \in C^1$  is convex then such minimizer  $u$  is a stationary point  $F'|_u = 0$

# Variational inequality

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- Find  $u \in \mathbb{V}$  s.t.  $\forall v \in \mathbb{V} : F(u) \leq F(v)$
- If  $F \in C^1$  is convex then such minimizer  $u$  is a stationary point  $F'|_u = 0$
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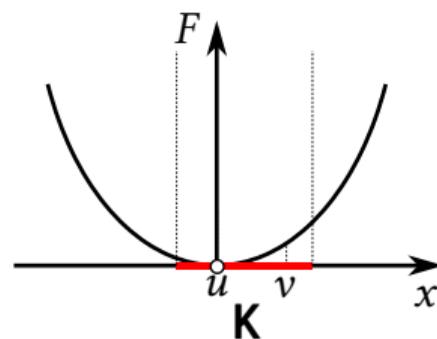
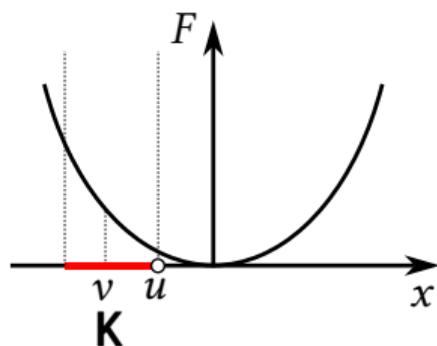
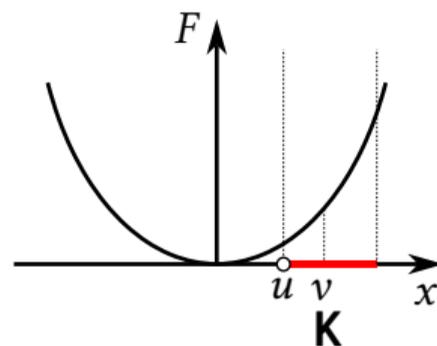
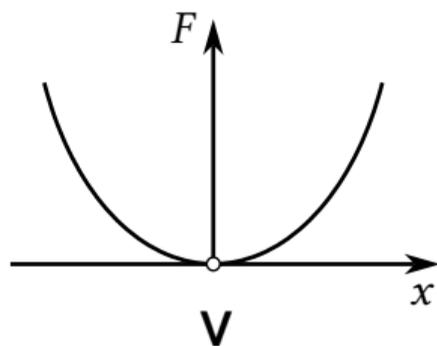
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- Variational inequality for minimizer  $u \in \mathbb{K} \subset \mathbb{V}$ :

$$F'(u)(v - u) \geq 0, \quad \forall v \in \mathbb{K}$$

# Example of variational inequality



Minimize  $F(x)$  for  $x \in \mathbb{K} \subset \mathbb{R}$ , then the minimizer  $u$  satisfies

$$F'(u)(v - u) \geq 0, \quad \forall v \in \mathbb{K}$$

# Variational inequality and a simplification

- Constrained minimization problem (variational inequality)<sup>[1,2]</sup>

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\bar{\Gamma}_c^1} \underline{\underline{\varrho}}_i^T \delta \xi_{\tilde{z}} d\bar{\Gamma}_c^1 \geq \int_{\bar{\Gamma}_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega, \quad \underline{\underline{u}} \in \mathbb{L}, \delta \underline{\underline{u}} \in \mathbb{K}$$

$$\mathbb{L} = \left\{ \underline{\underline{u}} \in H^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u, g_n(\underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

$$\mathbb{K} = \left\{ \delta \underline{\underline{u}} \in H^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u, g_n(\delta \underline{\underline{u}}) \geq 0 \text{ on } \Gamma_c \right\}$$

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- Use optimization theory to convert to

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\Gamma_c^1} \underbrace{C(\sigma_n, \sigma_t, g_n, \underline{\underline{\xi}}, \delta \underline{\underline{u}})}_{\text{Contact term}^*} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

Unconstrained functional sub-spaces

$$\mathbb{L} = \{ \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u \}$$

$$\mathbb{K} = \{ \delta \underline{\underline{u}} \in \mathbb{H}^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \}$$

Contact term\* is defined on the *potential contact zone*  $\Gamma_c^1$ .

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# Optimization methods

# Optimization methods

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

- Penalty method
- Lagrange multipliers method
- Augmented Lagrangian method

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

## ■ Penalty method

- New functional

$$F_p(\mathbf{x}) = F(\mathbf{x}) + \boxed{\epsilon \langle -g(\mathbf{x}) \rangle^2} = F(\mathbf{x}) + \begin{cases} 0, & \text{if } g(\mathbf{x}) \geq 0 & \text{non-contact} \\ \epsilon g^2(\mathbf{x}), & \text{if } g(\mathbf{x}) < 0 & \text{contact} \end{cases}$$

where  $\epsilon$  is the penalty parameter.

- Stationary point must satisfy

$$\nabla F_p(\mathbf{x}) = \nabla F(\mathbf{x}) + 2\epsilon \langle -g(\mathbf{x}) \rangle \nabla g(\mathbf{x}) = 0$$

- Solution **tends** to the precise solution as  $\epsilon \rightarrow \infty$

## ■ Lagrange multipliers method

## ■ Augmented Lagrangian method

$$\text{Macaulay brackets } \langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$$

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

■ Penalty method  $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$

## ■ Lagrange multipliers method

- New functional called **Lagrangian**

$$\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$$

- Saddle point problem

$$\min_x \max_\lambda \{\mathcal{L}(\mathbf{x}, \lambda)\} \longrightarrow \mathbf{x}^* \longleftarrow \min_{g(\mathbf{x}) \geq 0} \{F(\mathbf{x})\}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L} = \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0 \quad \text{need to verify } \lambda \leq 0$$

## ■ Augmented Lagrangian method

Macaulay brackets  $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

- Penalty method  $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$
- Lagrange multipliers method  $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$

## ■ Augmented Lagrangian method

[Hestnes 1969], [Powell 1969], [Glowinski & Le Tallec 1989], [Alart & Curnier 1991], [Simo & Laursen 1992]

- New functional, augmented Lagrangian

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \begin{cases} \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), & \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(\mathbf{x}) > 0, \text{ non-contact} \end{cases}$$

- Stationary point

$$\nabla_{\mathbf{x}, \lambda} \mathcal{L}_a = \begin{cases} \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) + \lambda \nabla_{\mathbf{x}} g(\mathbf{x}) + 2\epsilon g(\mathbf{x}) \nabla g(\mathbf{x}) \\ g(\mathbf{x}) \end{bmatrix} = 0, & \text{if contact} \\ \begin{bmatrix} \nabla_{\mathbf{x}} F(\mathbf{x}) \\ -\frac{\lambda}{\epsilon} \end{bmatrix} = 0, & \text{if non-contact} \end{cases}$$

Macaulay brackets  $\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} = \max\{x, 0\} = \text{ReLU}(x)$



Uzawa algorithm

Functional to be minimized  $F(\mathbf{x})$  under constraint  $g(\mathbf{x}) \geq 0$

- Penalty method  $F_p(\mathbf{x}) = F(\mathbf{x}) + \epsilon \langle -g(\mathbf{x}) \rangle^2$
- Lagrange multipliers method  $\mathcal{L}(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x})$
- **Augmented Lagrangian method: Uzawa algorithm**

Iterative procedure for  $\lambda$  update:

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + \lambda_i g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \text{ if } \lambda_i + 2\epsilon g(\mathbf{x}) \leq 0$$

$$\mathcal{L}_a(\mathbf{x}; \lambda_i) = F(\mathbf{x}) + (\lambda_i + \epsilon g(\mathbf{x})) g(\mathbf{x}), \text{ if } \lambda_i + 2\epsilon g(\mathbf{x}) > 0$$

$$\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$$

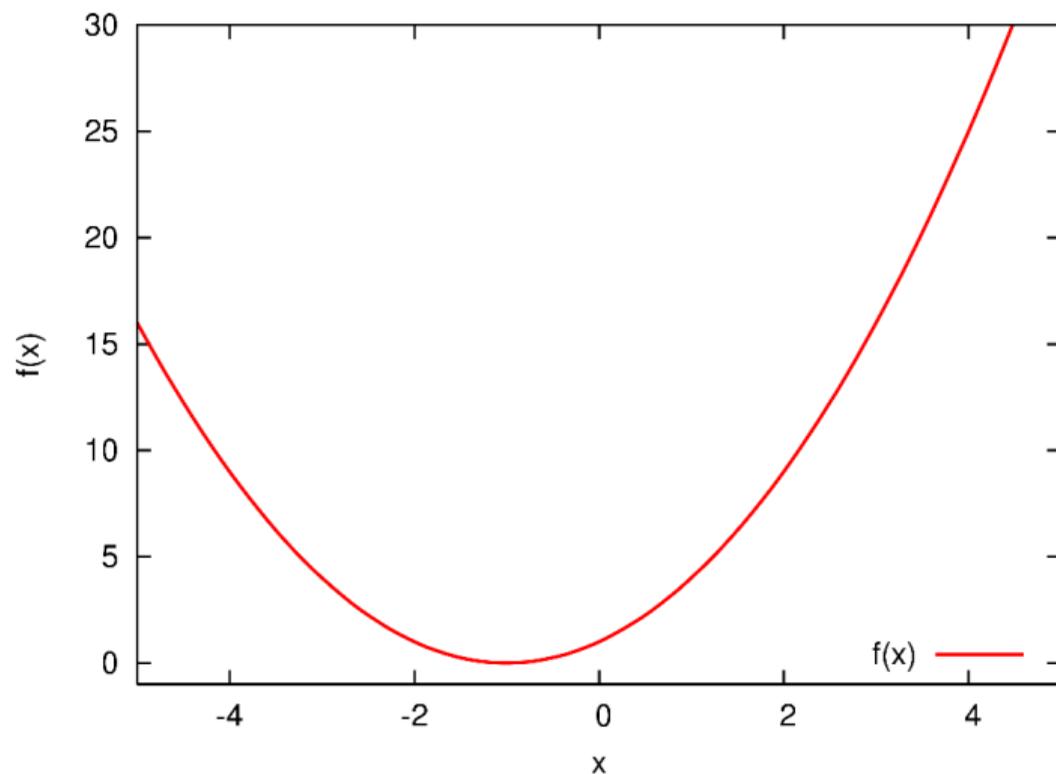
$$\frac{\partial \mathcal{L}_a}{\partial \mathbf{x}} = \frac{\partial F}{\partial \mathbf{x}} + (\lambda_i + 2\epsilon g(\mathbf{x})) \frac{\partial g}{\partial \mathbf{x}} = 0$$

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Uzawa algorithm

# Optimization methods: example

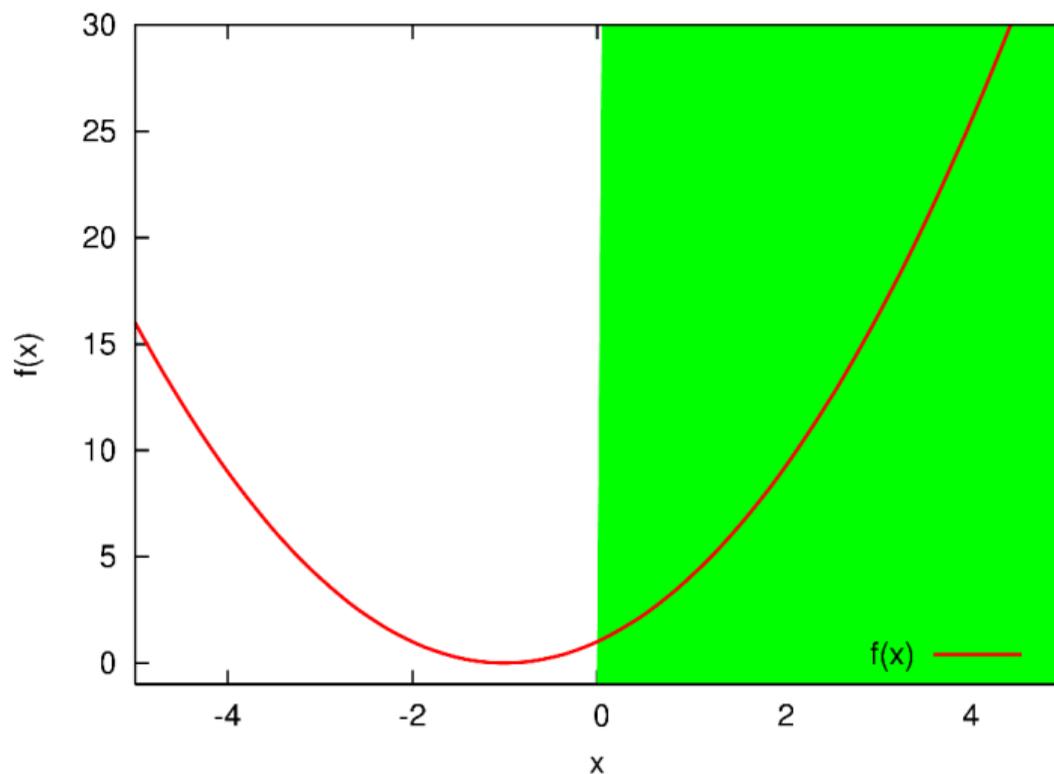


**Functional:**  $f(x) = x^2 + 2x + 1$

**Constraint:**  $g(x) = x \geq 0$

**Solution:**  $x^* = 0$

# Optimization methods: example



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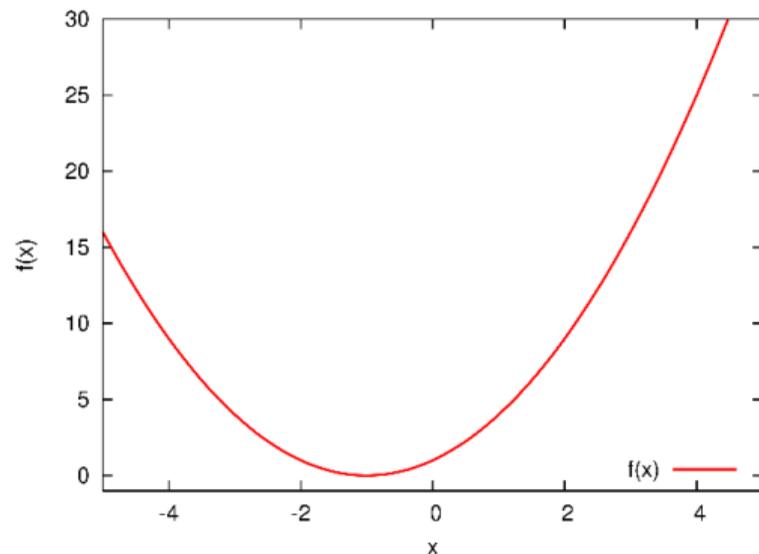
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# Penalty method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Penalty method

$$F_p(x) = F(x) + \epsilon \langle -g(x) \rangle^2$$

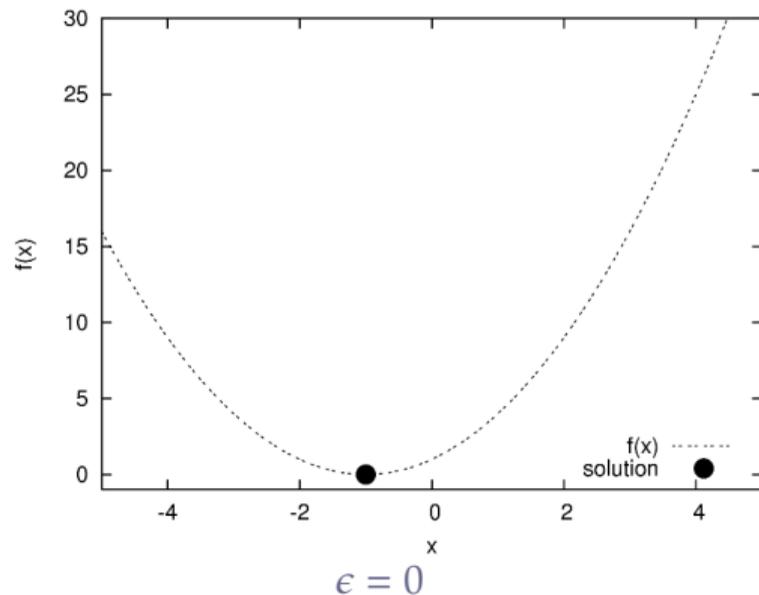
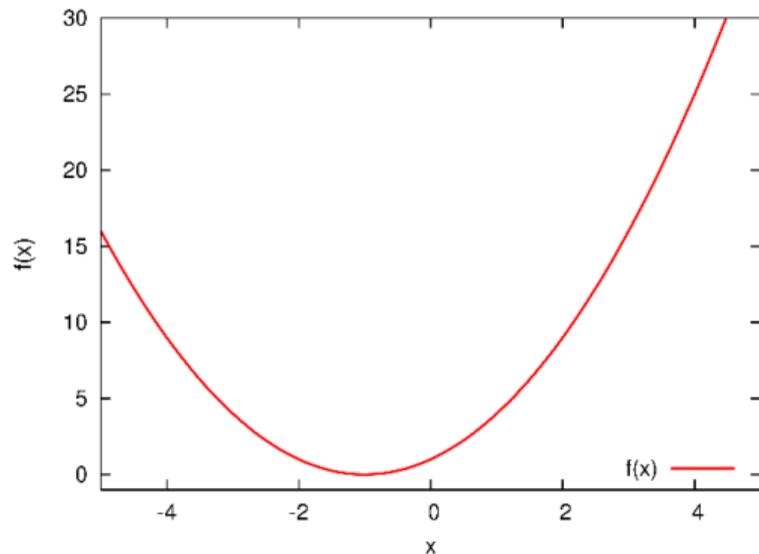


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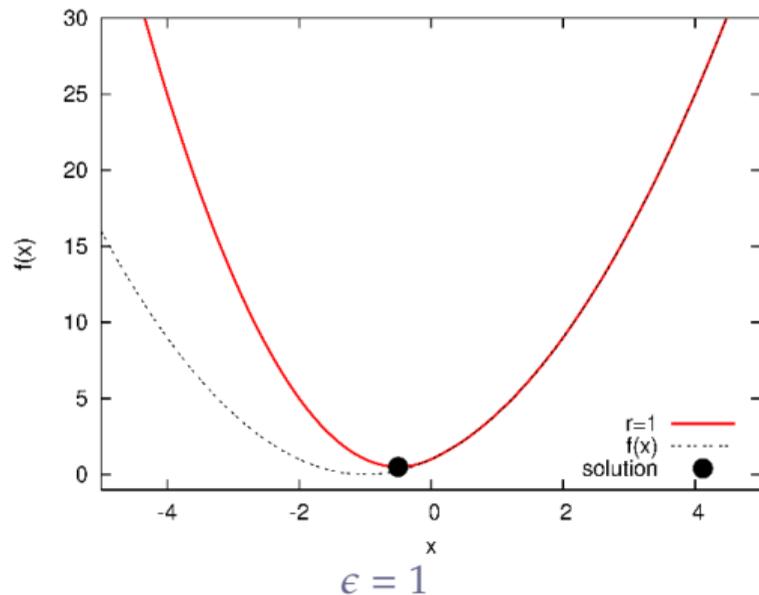
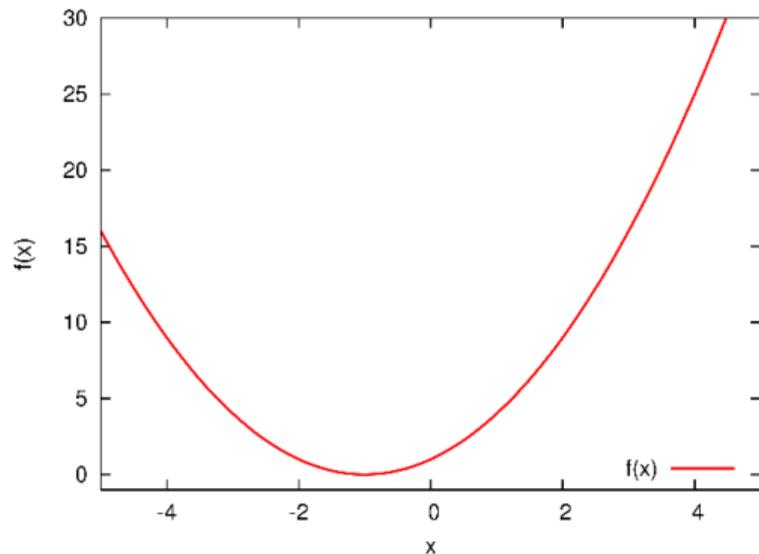


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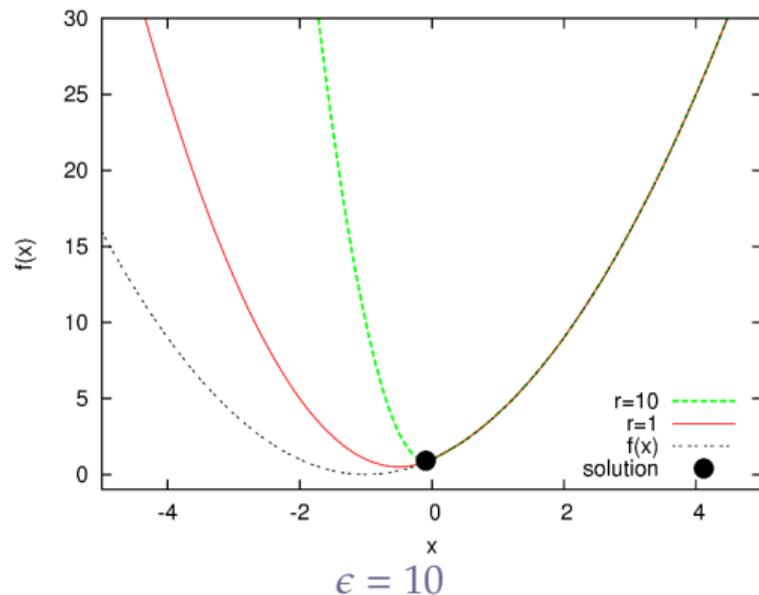
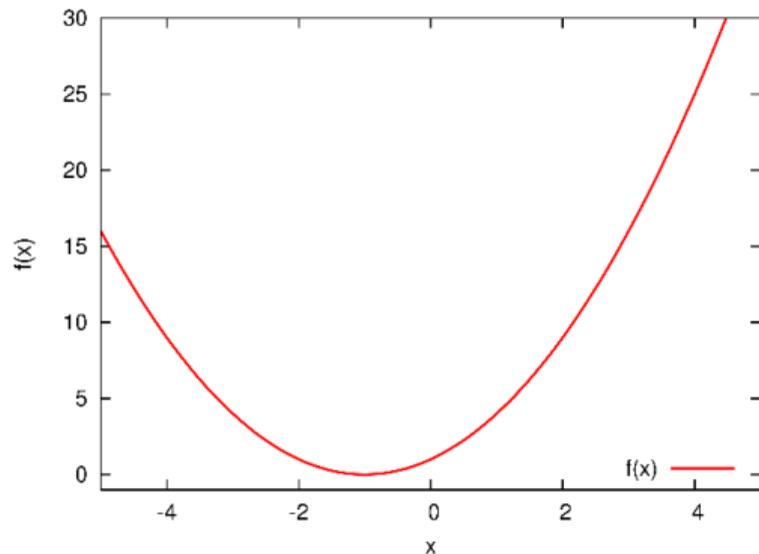


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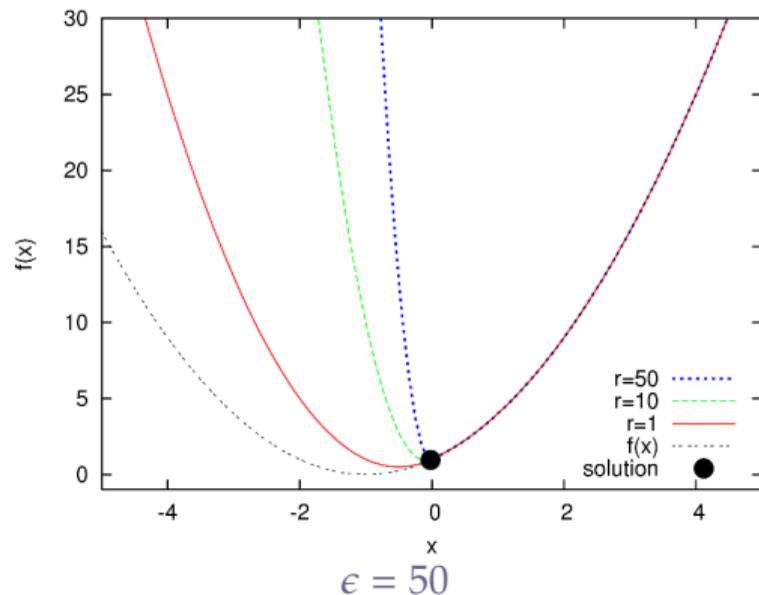
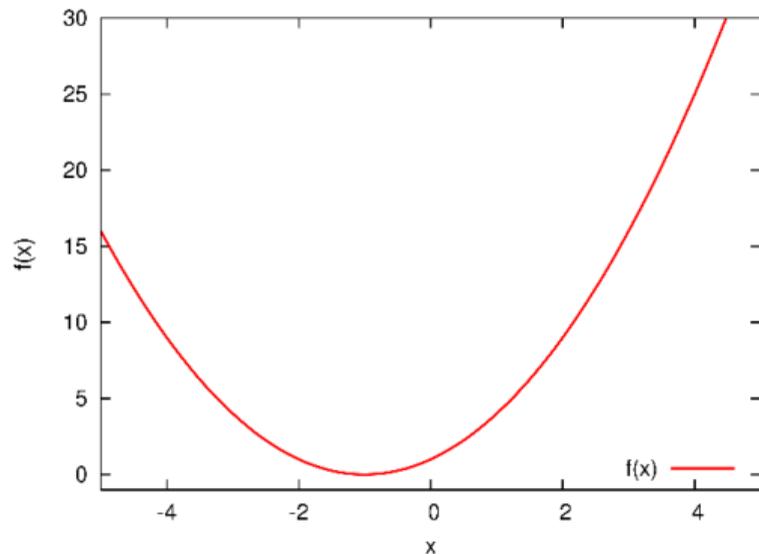


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### Advantages 😊

- simple physical interpretation
- simple implementation
- no additional degrees of freedom
- “mathematically” smooth functional

### Drawbacks 😞

- practically non-smooth functional
- solution is not exact:
  - too small penalty  $\rightarrow$  large penetration
  - too large penalty  $\rightarrow$  ill-conditioning of the tangent matrix
- user has to choose penalty  $\epsilon$  properly or automatically and/or adapt during convergence

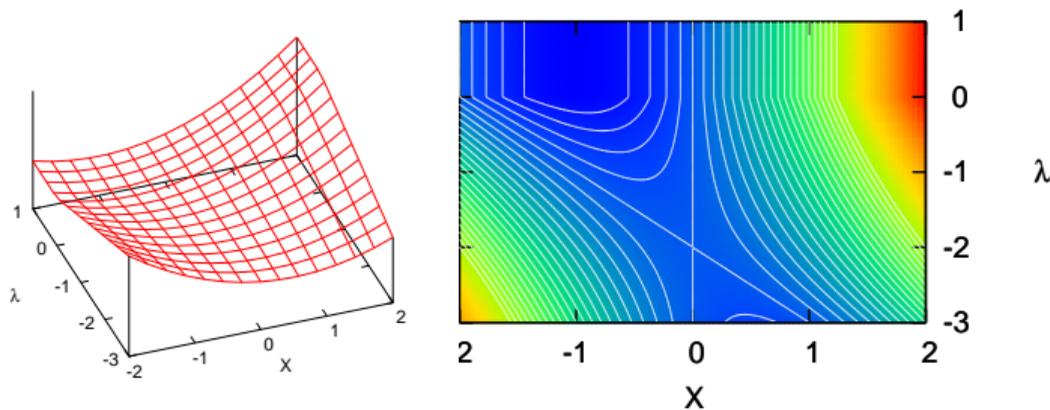
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## ■ Lagrange multipliers method

$$\mathcal{L}(x, \lambda) = F(x) + \boxed{\lambda g(x)} \rightarrow \text{Saddle point} \rightarrow \min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

Need to check that  $\lambda \leq 0$



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## Advantages 😊

- exact solution
- no adjustable parameters

## Drawbacks ☹️

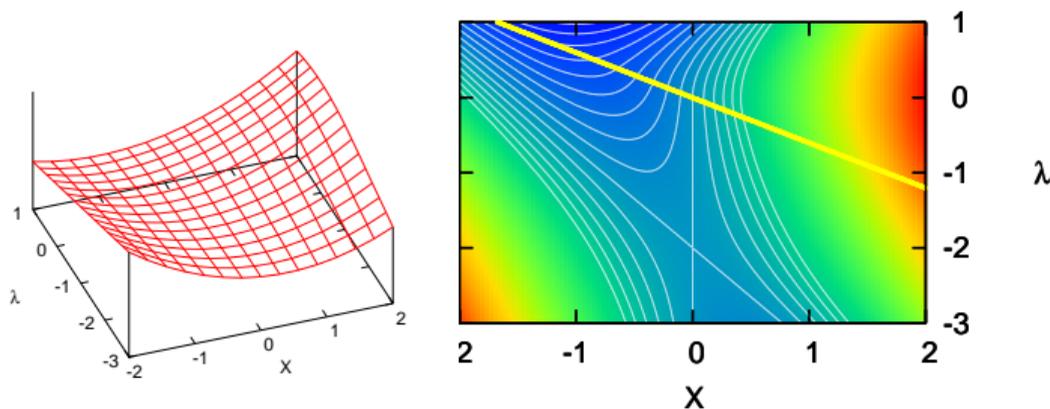
- Lagrangian is not smooth
- additional degrees of freedom
- not fully unconstrained:  $\lambda \leq 0$

# Augmented Lagrangian method: example

$$F(x) = x^2 + 2x + 1, \quad g(x) = x \geq 0, \quad x^* = 0$$

## ■ Augmented Lagrangian method

$$\mathcal{L}_a(x, \lambda) = F(x) + \begin{cases} \lambda g(x) + \epsilon g^2(x), & \text{if } \lambda + 2\epsilon g(x) \leq 0, \text{ contact} \\ -\frac{1}{4\epsilon} \lambda^2, & \text{if } \lambda + 2\epsilon g(x) > 0, \text{ non-contact} \end{cases}$$



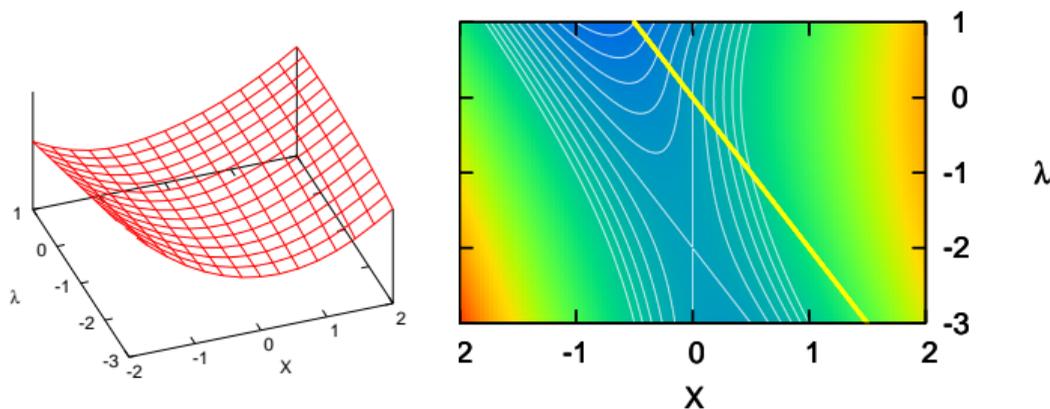
Yellow line separates contact and non-contact regions

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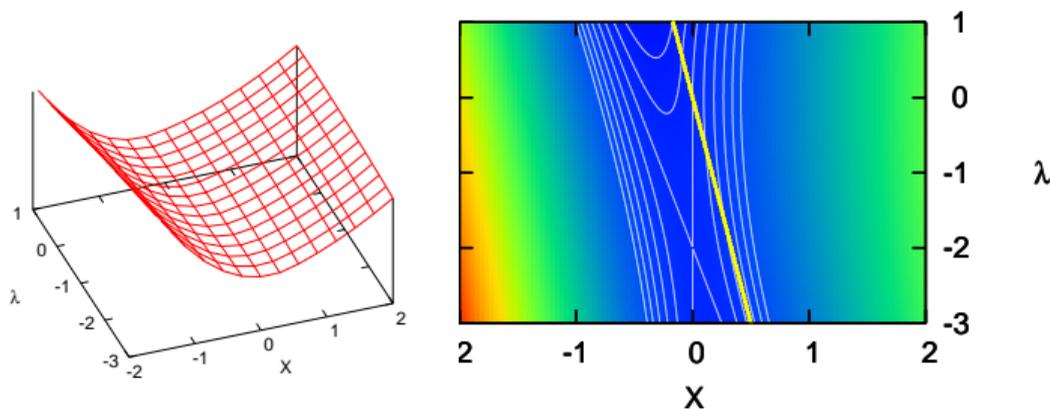
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## Advantages ☺

- exact solution
- smoother functional (!)
- fully unconstrained

## Drawbacks ☹

- additional degrees of freedom
- quite sensitive to parameter  $\epsilon$
- need to adjust  $\epsilon$  during convergence

## ■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}; \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \text{ contact}$$

Fix  $\lambda = \lambda_0$

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + \lambda_0 g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \quad \text{if } \lambda_0 + 2\epsilon g(\mathbf{x}) \leq 0$$

Converge with respect to  $x$

# Augmented Lagrangian with Uzawa algorithm

## ■ Augmented Lagrangian method

$$\mathcal{L}_a(\mathbf{x}; \lambda) = F(\mathbf{x}) + \lambda g(\mathbf{x}) + \epsilon g^2(\mathbf{x}), \quad \text{if } \lambda + 2\epsilon g(\mathbf{x}) \leq 0, \text{ contact}$$

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Converge with respect to  $x$  and update  $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

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Fix  $\lambda = \lambda_0$

Converge with respect to  $x$  and update  $\lambda_{i+1} = \lambda_i + \epsilon g(\mathbf{x})$

$$\mathcal{L}_a(\mathbf{x}, \lambda) = F(\mathbf{x}) + [\lambda_1 + \epsilon g(\mathbf{x})] g(\mathbf{x}), \quad \text{if } \lambda_1 + 2\epsilon g(\mathbf{x}) \leq 0$$

# Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\Gamma_c^1} \underbrace{C}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

$$\underline{\underline{u}} \in \mathbb{L}, \delta \underline{\underline{u}} \in \mathbb{K}, \quad \mathbb{L} = \{ \underline{\underline{u}} \in H^1(\Omega) \mid \underline{\underline{u}} = \underline{\underline{u}}_0 \text{ on } \Gamma_u \}, \quad \mathbb{K} = \{ \delta \underline{\underline{u}} \in H^1(\Omega) \mid \delta \underline{\underline{u}} = 0 \text{ on } \Gamma_u \}$$

## ■ Penalty method

$$\text{Pressure: } \sigma_n = \epsilon g_n, \quad \text{Shear: } \underline{\underline{\sigma}}_t = \begin{cases} \epsilon \underline{\underline{g}}_t', & \text{if stick } |\sigma_t| < \mu |\sigma_n| \\ \mu \epsilon g_n \delta \underline{\underline{g}}_t / |\delta \underline{\underline{g}}_t|, & \text{if slip } |\sigma_t| = \mu |\sigma_n| \end{cases}$$

Contact term

$$C = C(g_n, \underline{\underline{g}}_t, \delta g_n, \delta \underline{\underline{g}}_t) = \sigma_n \delta g_n + \underline{\underline{\sigma}}_t \cdot \delta \underline{\underline{g}}_t$$

# Application to contact problems: weak form

$$\int_{\Omega} \underline{\underline{\sigma}} \cdot \delta \nabla \underline{\underline{u}} d\Omega + \int_{\Gamma_c^1} \underbrace{\boxed{C}}_{\text{Contact term}} d\Gamma_c^1 = \int_{\Gamma_f} \underline{\underline{\sigma}}_0 \cdot \delta \underline{\underline{u}} d\Gamma + \int_{\Omega} \underline{\underline{f}}_v \cdot \delta \underline{\underline{u}} d\Omega,$$

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## ■ Augmented Lagrangian method

Contact term

$$C = C(g_n, \underline{\underline{g}}_t, \lambda_n, \underline{\underline{\lambda}}_t, \delta g_n, \delta \underline{\underline{g}}_t, \delta \lambda_n, \delta \underline{\underline{\lambda}}_t)$$

$$C = \begin{cases} -\frac{1}{\epsilon} (\lambda_n \delta \lambda_n - \underline{\underline{\lambda}}_t \cdot \delta \underline{\underline{\lambda}}_t), & \text{if non-contact } \lambda_n + \epsilon g_n \geq 0 \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \underline{\underline{\lambda}}_t \cdot \delta \underline{\underline{g}}_t + \underline{\underline{g}}_t \cdot \delta \underline{\underline{\lambda}}_t, & \text{if stick } |\underline{\underline{\lambda}}_t| \leq \mu |\hat{\sigma}_n| \\ \hat{\lambda}_n \delta g_n + g_n \delta \lambda_n + \mu \hat{\sigma}_n - \mu \hat{\sigma}_n \frac{\underline{\underline{\lambda}}_t}{|\underline{\underline{\lambda}}_t|} \cdot \delta \underline{\underline{g}}_t - \frac{1}{\epsilon} \left( \lambda_t + \mu \hat{\sigma}_n \frac{\underline{\underline{\lambda}}_t}{|\underline{\underline{\lambda}}_t|} \right) \cdot \delta \underline{\underline{\lambda}}_t, & \text{if slip } |\underline{\underline{\lambda}}_t| \geq \mu |\hat{\sigma}_n| \end{cases}$$

where  $\hat{\lambda}_n = \lambda_n + \epsilon g_n$  and  $\underline{\underline{\lambda}}_t = \underline{\underline{\lambda}}_t + \epsilon \underline{\underline{g}}_t$ .

# Application to contact problems: linearization

- Non-linear equation

$$R(\underline{u}, \underline{f}) = 0$$

- Contains  $\delta g_n, \delta g_t$
- Use Newton-Raphson method
- Initial state at step  $i$

$$R(\underline{u}^i, \underline{f}^i) = 0$$

- Should be also satisfied at step  $i + 1$

$$R(\underline{u}^{i+1}, \underline{f}^{i+1}) = R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = 0$$

- Linearize

$$R(\underline{u}^i + \delta \underline{u}, \underline{f}^{i+1}) = R(\underline{u}^i, \underline{f}^{i+1}) + \frac{\partial R(\underline{u})}{\partial \underline{u}} \delta \underline{u} = 0$$

- Finally

$$\delta \underline{u} = - \underbrace{\left[ \frac{\partial R(\underline{u})}{\partial \underline{u}} \right]^{-1}}_{\text{contains } \Delta \delta g_n, \Delta \delta g_t} R(\underline{u}^i)$$

- NB: Contact problem does not satisfy conditions of Kantorovich theorem on the convergence of Newton's method.

## Normal gap

- First variation enters in the residual vector:

$$\delta g_n = \underline{n} \cdot (\delta \underline{r}_s - \delta \underline{\rho})$$

- Second variation enters in the tangent matrix:

$$\begin{aligned} \Delta \delta g_n = & - \underline{n} \cdot \left( \delta \frac{\partial \underline{\rho}^T}{\partial \underline{\xi}} \Delta \underline{\xi} + \Delta \frac{\partial \underline{\rho}^T}{\partial \underline{\xi}} \delta \underline{\xi} \right) - \Delta \underline{\xi}^T \underline{\underline{H}} \delta \underline{\xi} + \\ & + g_n \left( \Delta \underline{\xi}^T \underline{\underline{H}} + \underline{n} \cdot \Delta \frac{\partial \underline{\rho}^T}{\partial \underline{\xi}} \right) \bar{\underline{A}} \left( \underline{n} \cdot \delta \frac{\partial \underline{\rho}}{\partial \underline{\xi}} + \underline{\underline{H}} \delta \underline{\xi} \right) \end{aligned}$$

## Convective coordinate of the projection

- First variation enters in the residual vector:

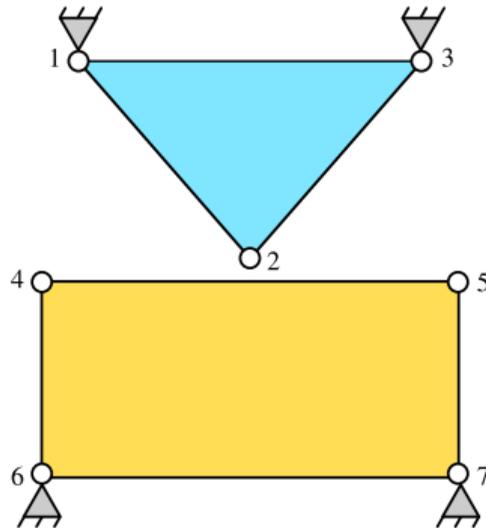
$$\delta_{\tilde{\xi}} \xi = \left[ \underline{\tilde{\mathbb{A}}} - g_n \underline{\tilde{\mathbb{H}}} \right]^{-1} \left( \frac{\partial \rho}{\partial \tilde{\xi}} \cdot (\delta \underline{r}_s - \delta \underline{\rho}) + g_n \underline{n} \cdot \delta \frac{\partial \rho}{\partial \tilde{\xi}} \right)$$

- Second variation enters in the tangent matrix:

$$\begin{aligned} \Delta \delta_{\tilde{\xi}} \xi &= (g_n \underline{\tilde{\mathbb{H}}} - \underline{\tilde{\mathbb{A}}})^{-1} \left\{ \frac{\partial \rho}{\partial \tilde{\xi}} \cdot \left( \delta \frac{\partial \rho^T}{\partial \tilde{\xi}} \Delta \tilde{\xi} + \Delta \frac{\partial \rho^T}{\partial \tilde{\xi}} \delta \tilde{\xi} \right) + \Delta \tilde{\xi}^T \left( \frac{\partial \rho}{\partial \tilde{\xi}} \cdot \frac{\partial^2 \rho}{\partial \tilde{\xi}^2} \right) \delta \tilde{\xi} - \right. \\ &\quad - g_n \underline{n} \cdot \left( \delta \frac{\partial^2 \rho}{\partial \tilde{\xi}^2} \Delta \tilde{\xi} + \Delta \frac{\partial^2 \rho}{\partial \tilde{\xi}^2} \delta \tilde{\xi} \right) - g_n \Delta \tilde{\xi}^T \left( \underline{n} \cdot \frac{\partial^3 \rho}{\partial \tilde{\xi}^3} \right) \delta \tilde{\xi} + \\ &\quad + \left[ g_n \left( \delta \frac{\partial \rho}{\partial \tilde{\xi}} + \frac{\partial^2 \rho}{\partial \tilde{\xi}^2} \delta \tilde{\xi} \right) \cdot \frac{\partial \rho^T}{\partial \tilde{\xi}} \underline{\tilde{\mathbb{A}}} - \delta g_n \underline{\mathbb{I}} \right] \left( \underline{n} \cdot \Delta \frac{\partial \rho}{\partial \tilde{\xi}} + \underline{\tilde{\mathbb{H}}} \Delta \tilde{\xi} \right) + \\ &\quad + \left[ g_n \left( \Delta \frac{\partial \rho}{\partial \tilde{\xi}} + \frac{\partial^2 \rho}{\partial \tilde{\xi}^2} \Delta \tilde{\xi} \right) \cdot \frac{\partial \rho^T}{\partial \tilde{\xi}} \underline{\tilde{\mathbb{A}}} - \Delta g_n \underline{\mathbb{I}} \right] \left( \underline{n} \cdot \delta \frac{\partial \rho}{\partial \tilde{\xi}} + \underline{\tilde{\mathbb{H}}} \delta \tilde{\xi} \right) \left. \right\} \end{aligned}$$

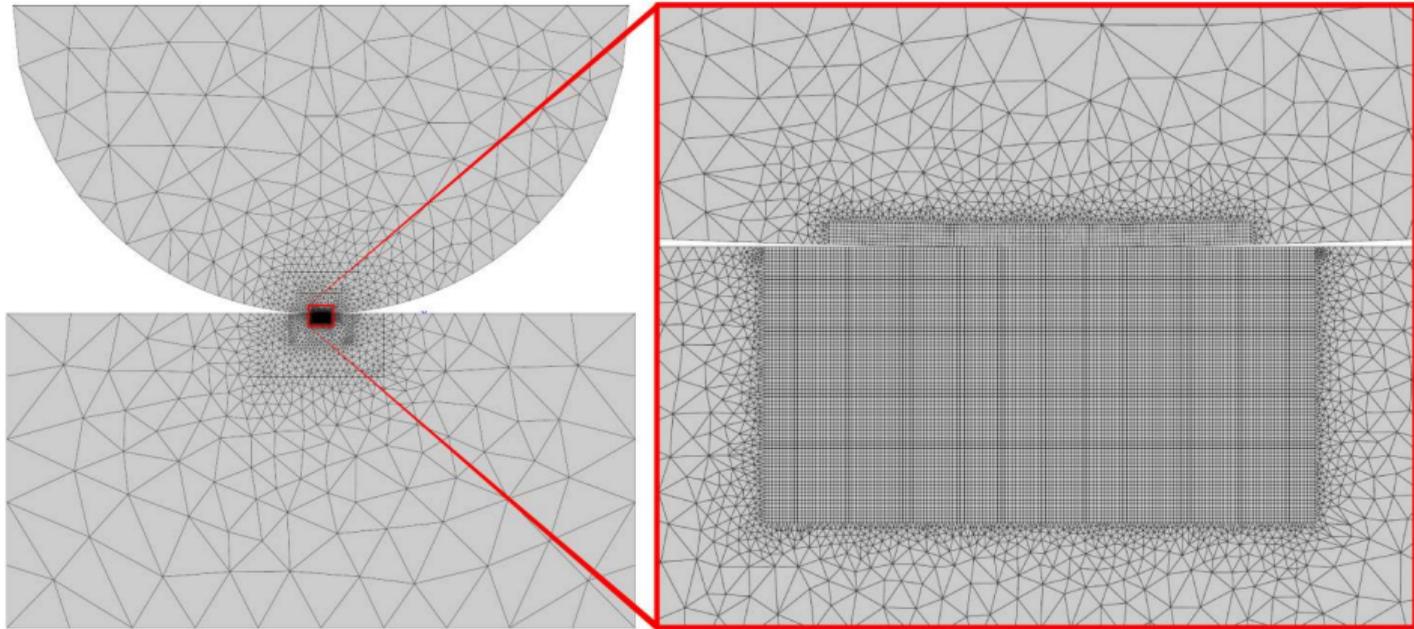
# Example

- Use penalty method to enforce Dirichlet BC
- Use penalty method to enforce contact constraints
- First, detect contact elements
- Second, construct updated residual vector and tangent matrix



# Particularities: mesh and convergence

- Strong **mesh refinement** is required
  - especially at **unknown edges** of contact zones

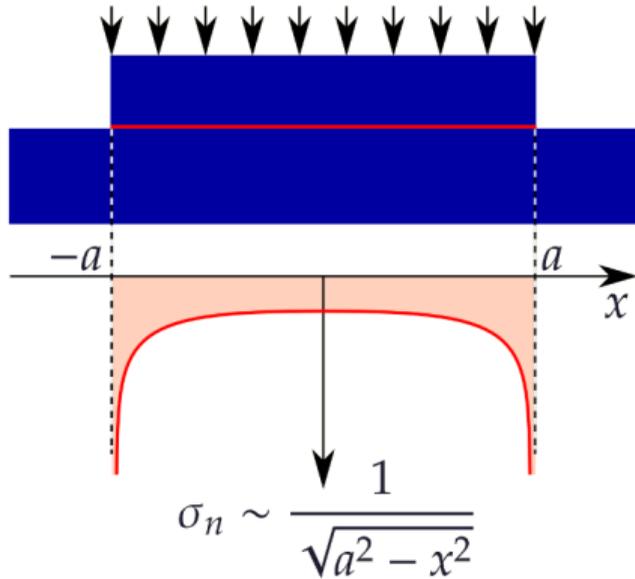


*Typical mesh for fretting analysis [L. Sun, H. Proudhon, G. Cailletaud, 2011]*

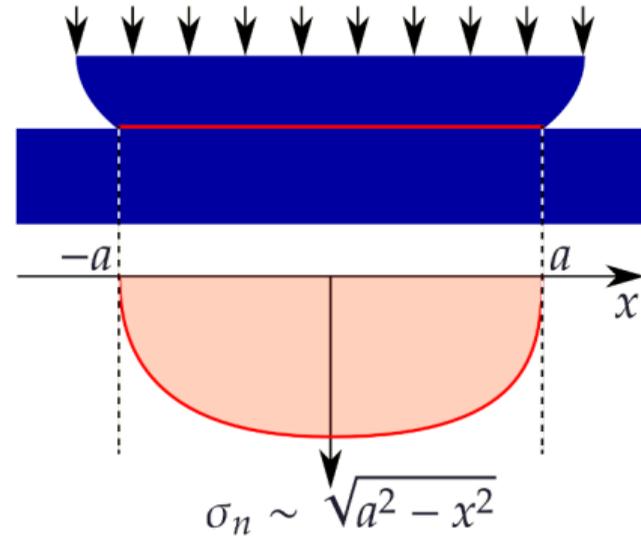
*2D ~ 30 000 DoFs, 3D ~ 5 000 000 DoFs*

# Particularities: mesh and convergence

- Strong **mesh refinement** is required
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$$\sigma_n \xrightarrow{x \rightarrow a} -\infty \quad \left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

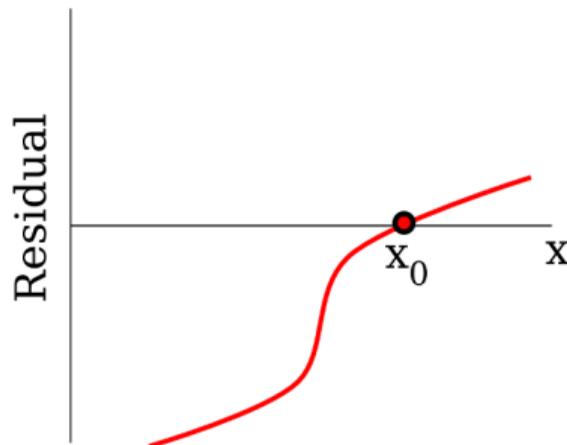


$$\left| \frac{\partial \sigma_n}{\partial x} \right| \xrightarrow{x \rightarrow a} \infty$$

*Infinite contact pressure and/or its derivative*

# Particularities: mesh and convergence

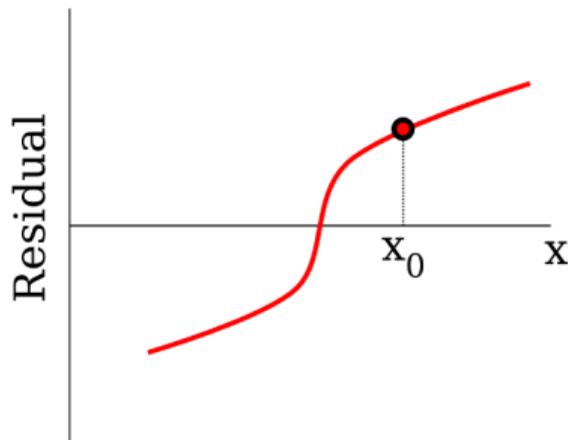
- Strong **mesh refinement** is required
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  - **Slow change** of boundary conditions:
    - strong non-linearities of contact/ friction problems
    - non-uniqueness of solution for frictional problems
- Infinite looping**



*Initial guess  $R(x_0, f_0) = 0$*

# Particularities: mesh and convergence

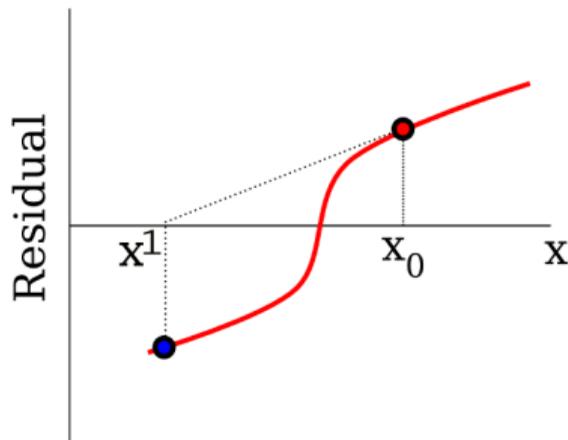
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*Too rapid change in boundary conditions  $R(x_0, f_1) \neq 0$*

# Particularities: mesh and convergence

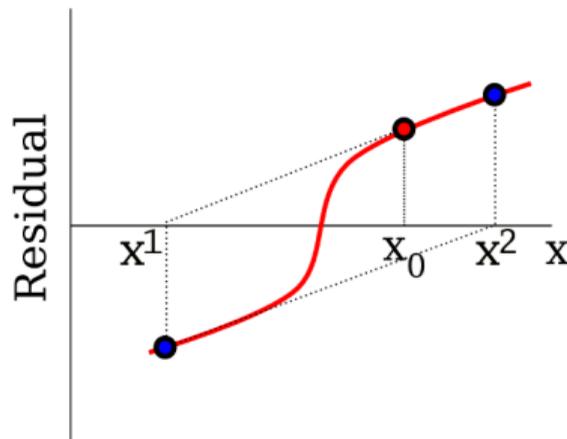
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$$\text{Iterations of Newton-Raphson method } R(x_0, f_1) + \left. \frac{\partial R}{\partial x} \right|_{x_0} \delta x = 0 \rightarrow \delta x = - \left. \frac{\partial R}{\partial x} \right|_{x_0}^{-1} R(x_0, f_1) \rightarrow x^1 = x_0 + \delta x$$

# Particularities: mesh and convergence

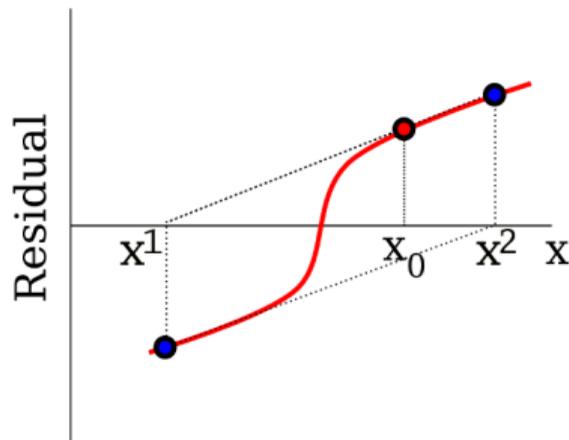
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# Particularities: mesh and convergence

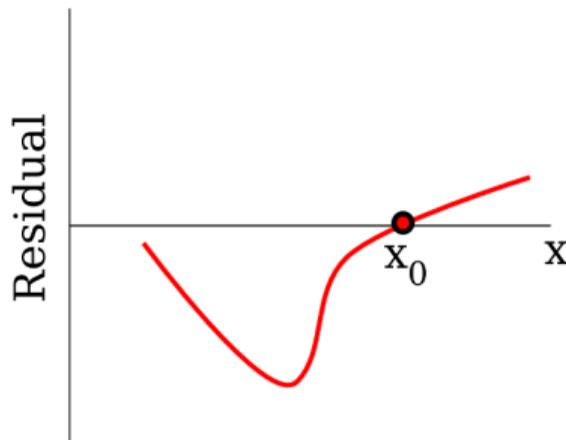
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*Infinite looping*

# Particularities: mesh and convergence

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- Convergence to a "false" solution

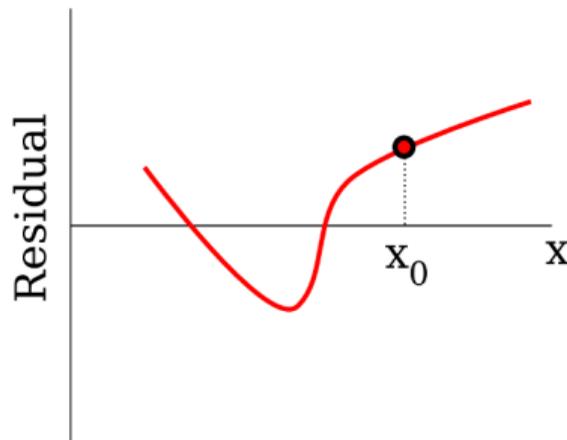


*Initial guess  $R(x_0, f_0) = 0$*

# Particularities: mesh and convergence

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Convergence to a "false" solution

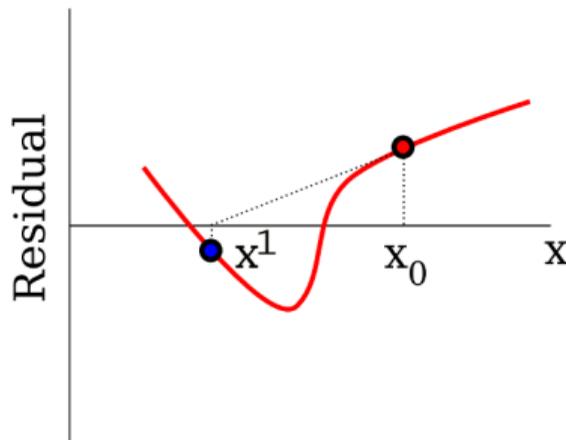


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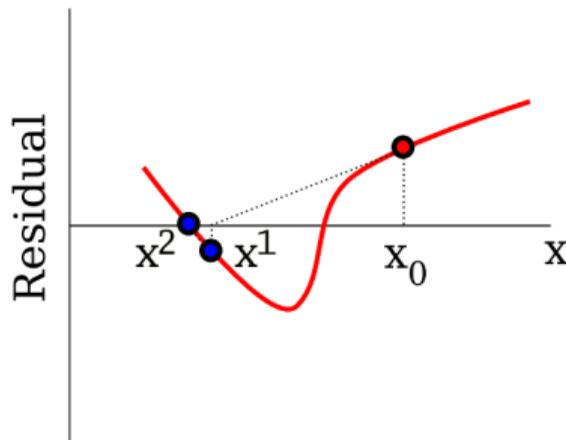


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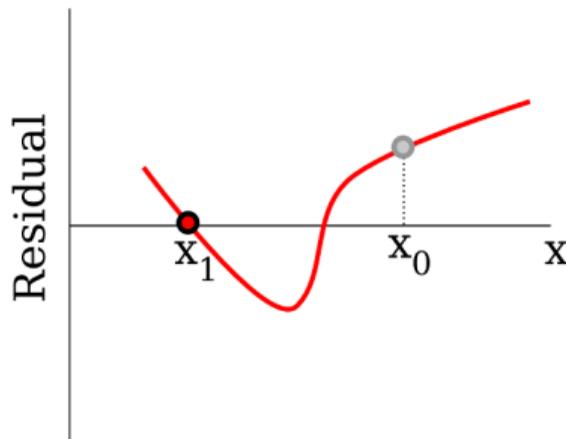


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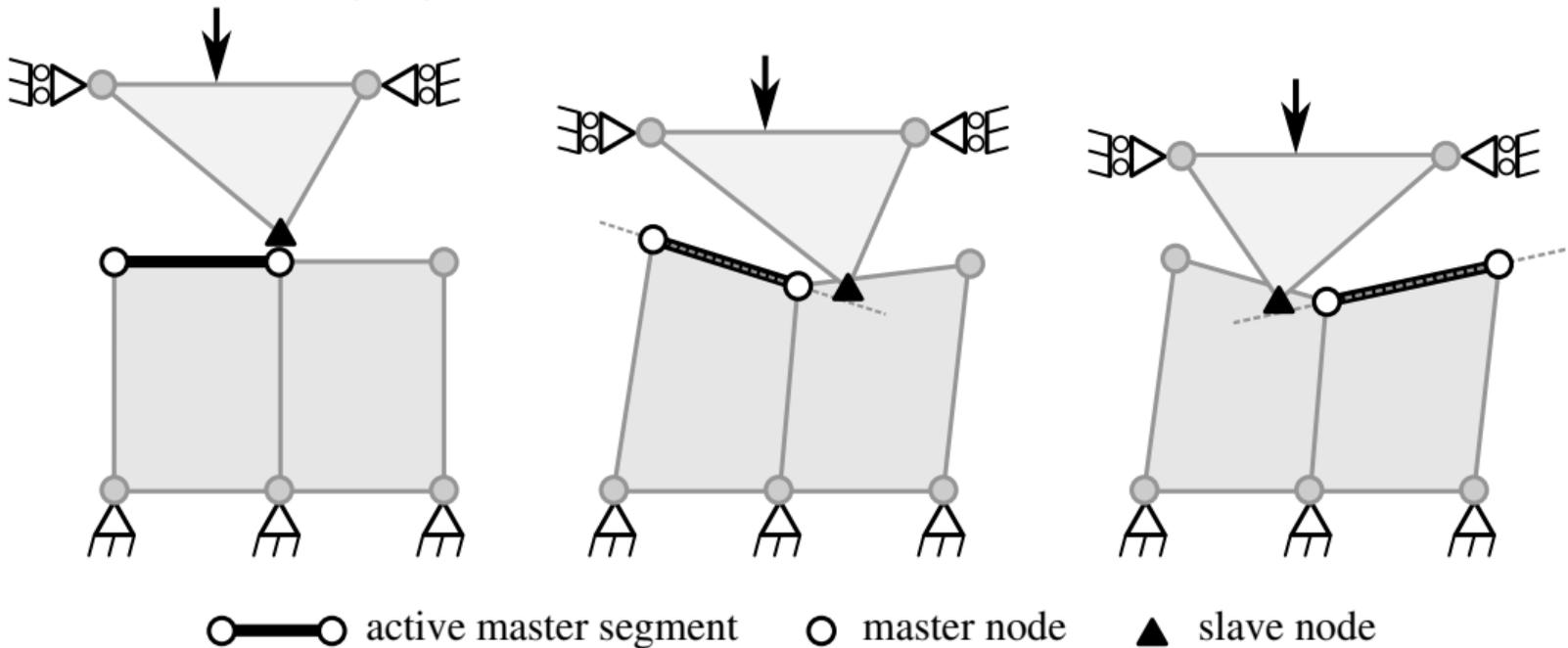
Convergence to a "false" solution



*Convergence, but is it a "true" solution ?*

# Convergence problems: examples

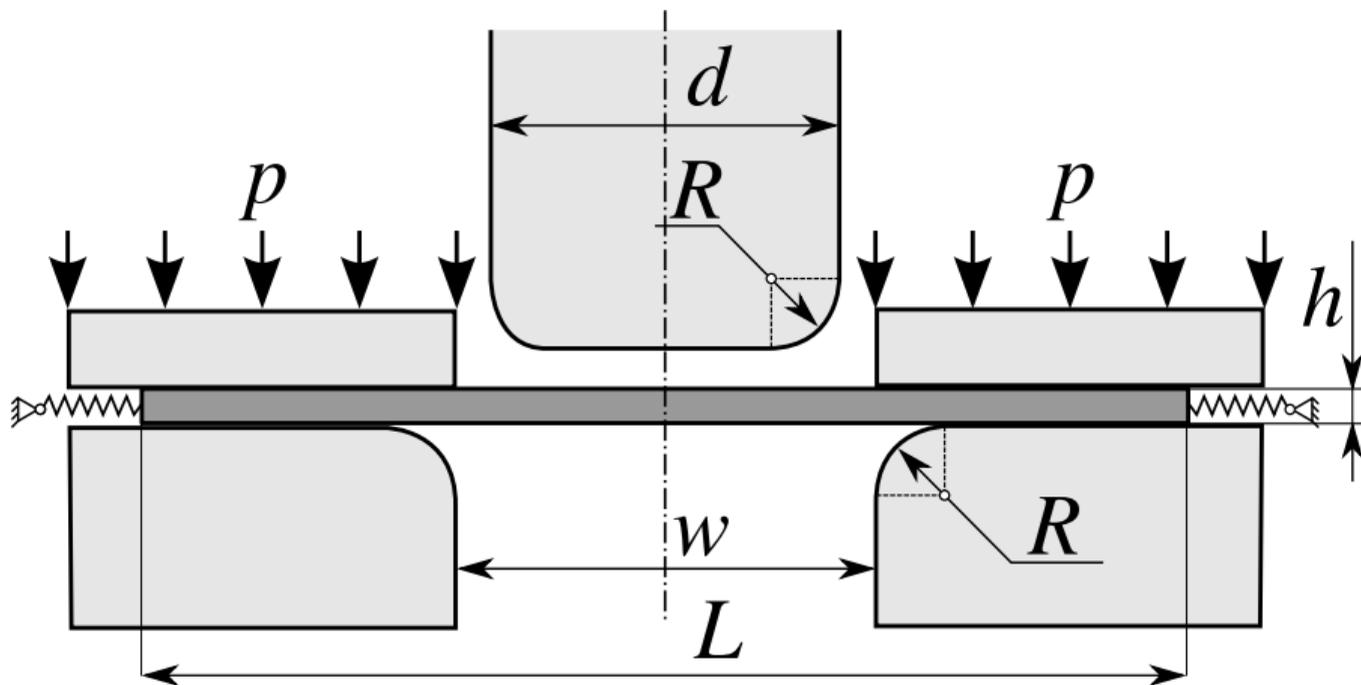
## ■ Infinite looping, e.g.



- Change of the contact state (contact/non-contact, stick/slip)
- Interplay between stiffness, friction and augmented Lagrangian coefficients<sup>[1]</sup>
- Combination of non-linearities (e.g., plasticity+contact)

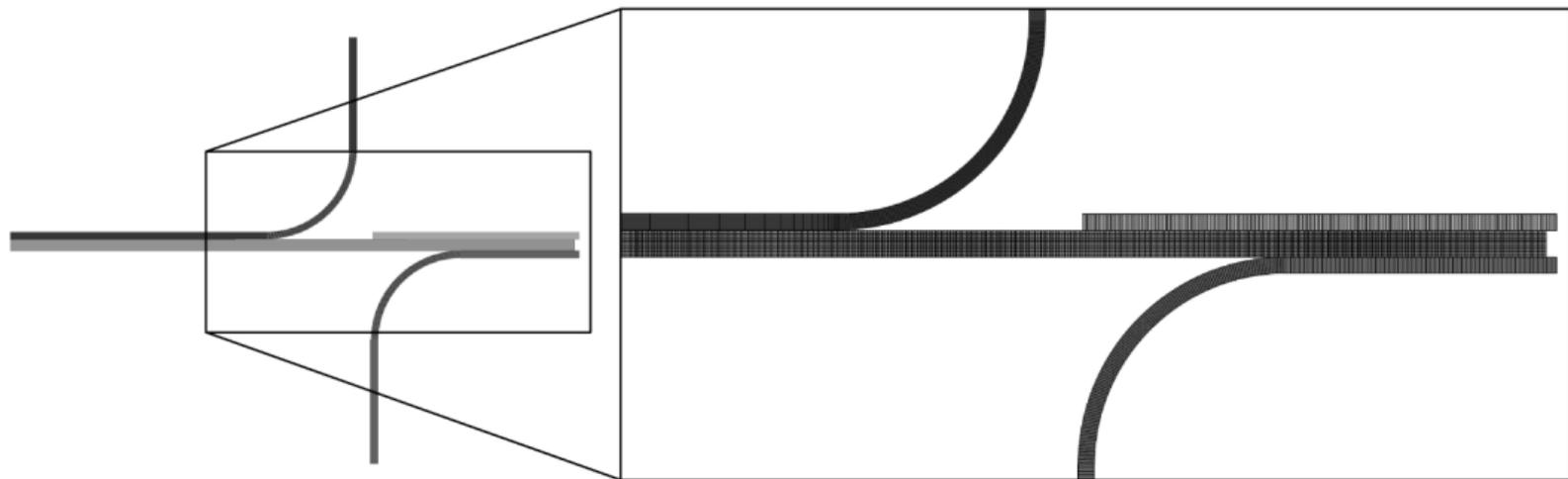
# Convergence problems: examples

- Simulation of a deep drawing problem
- Finite strain plasticity + frictional contact



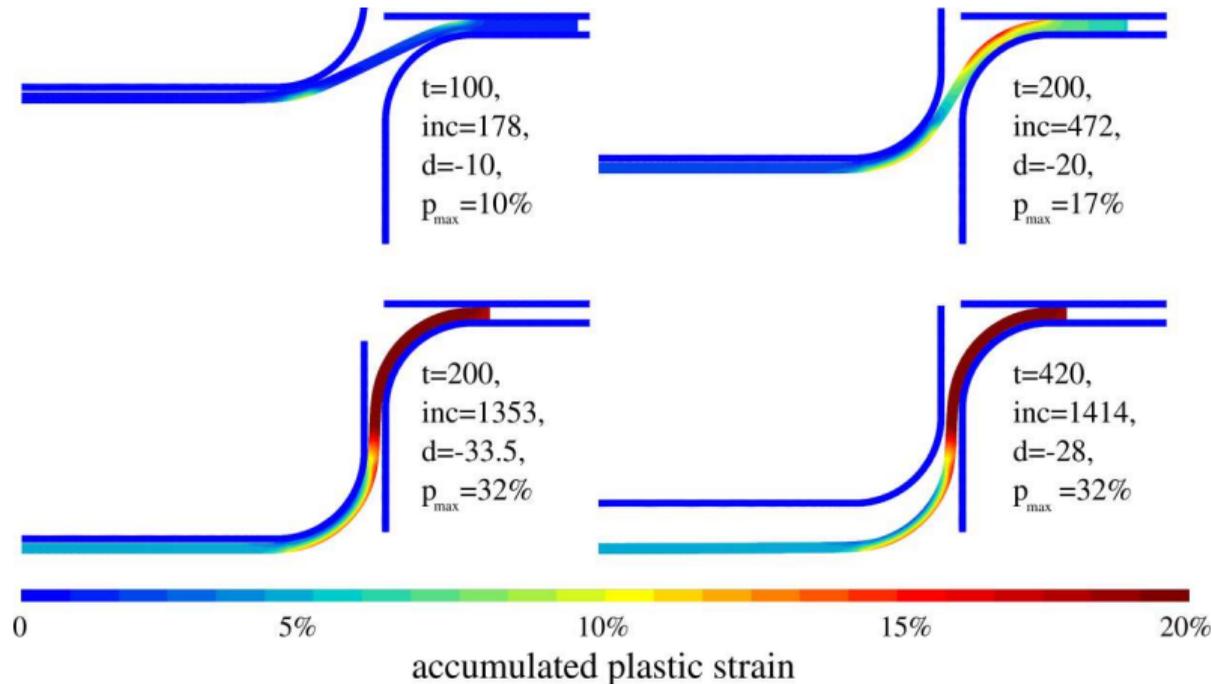
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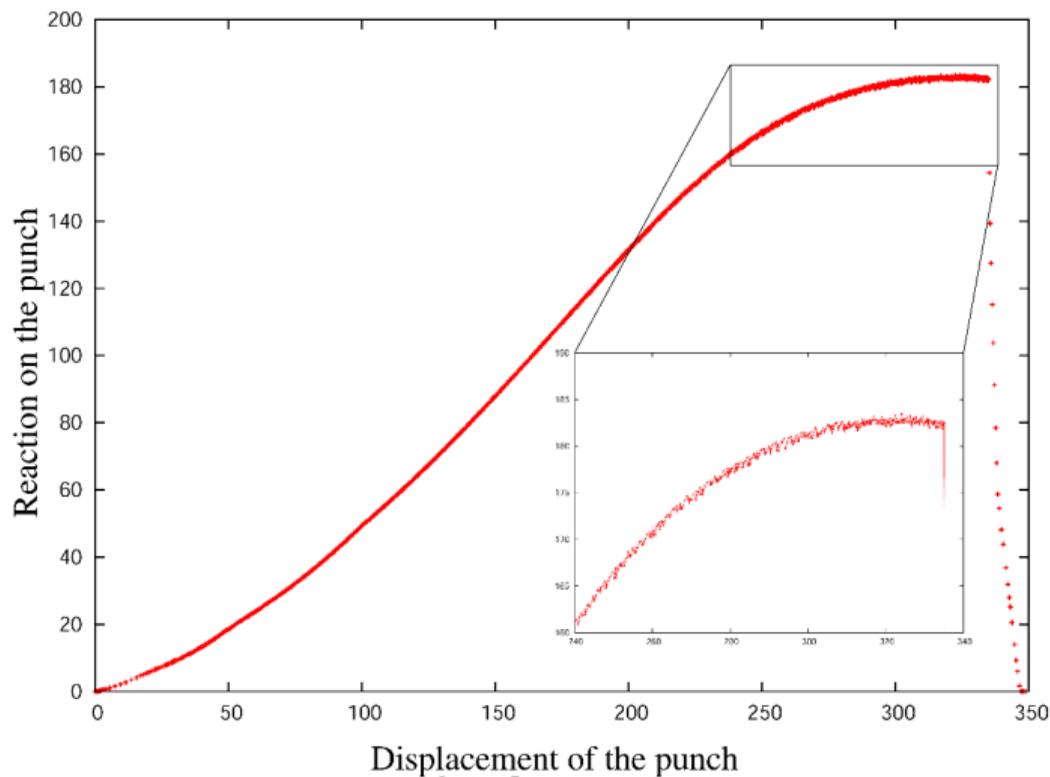
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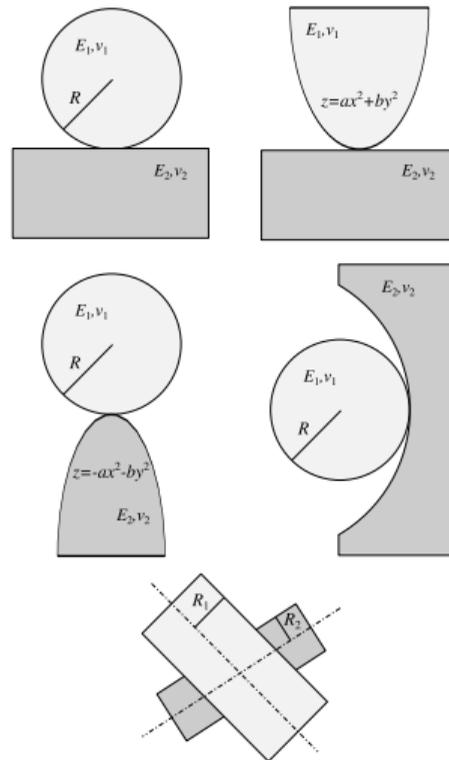
# Convergence problems: examples

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# Cylinder-plane frictional contact

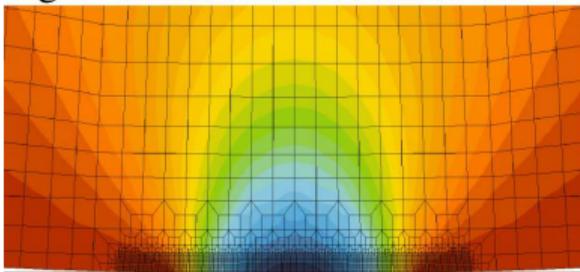
- Non-conservative problem, history of loading is crucial



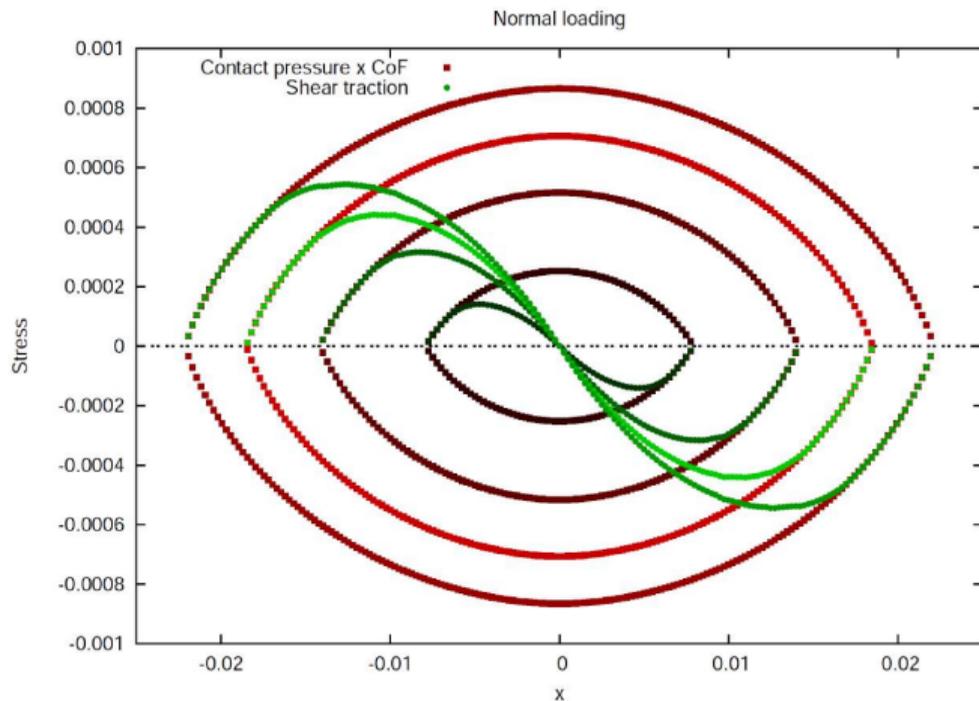
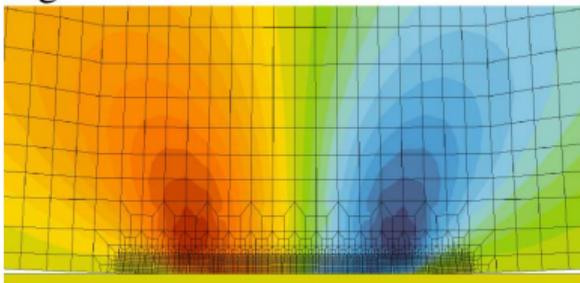
# Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal normal load



sig12 at maximal normal load

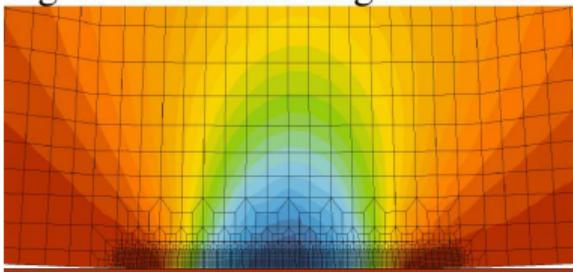


Press in 100 increments,  $u_z \sim t^2$

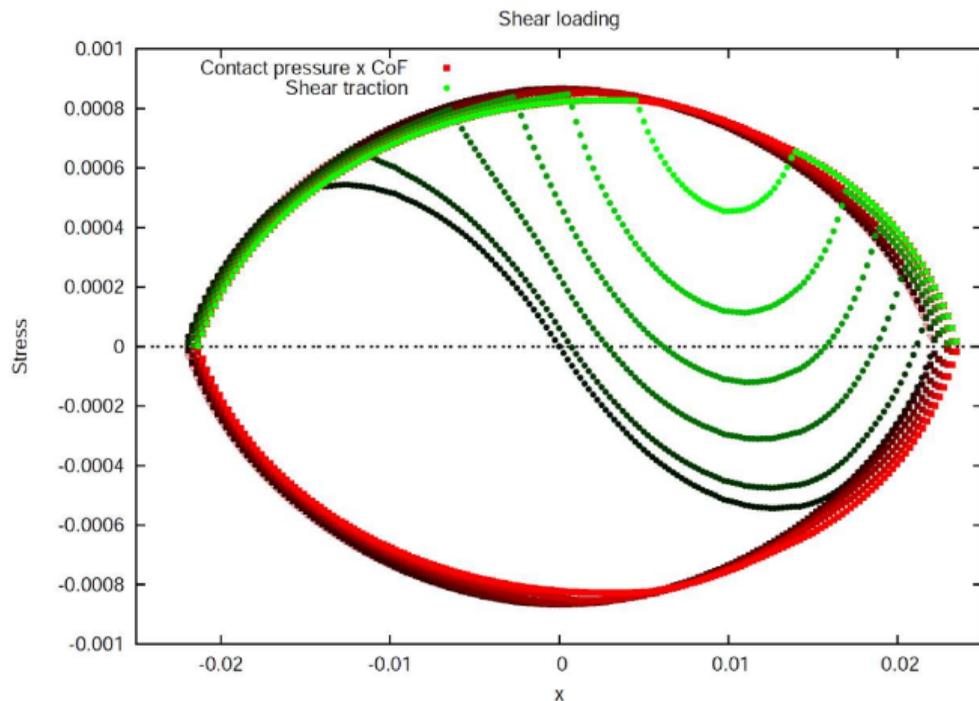
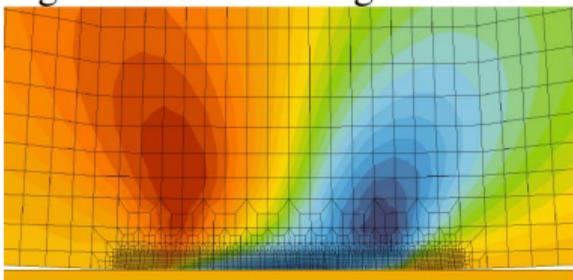
# Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

sig22 at maximal tangential load



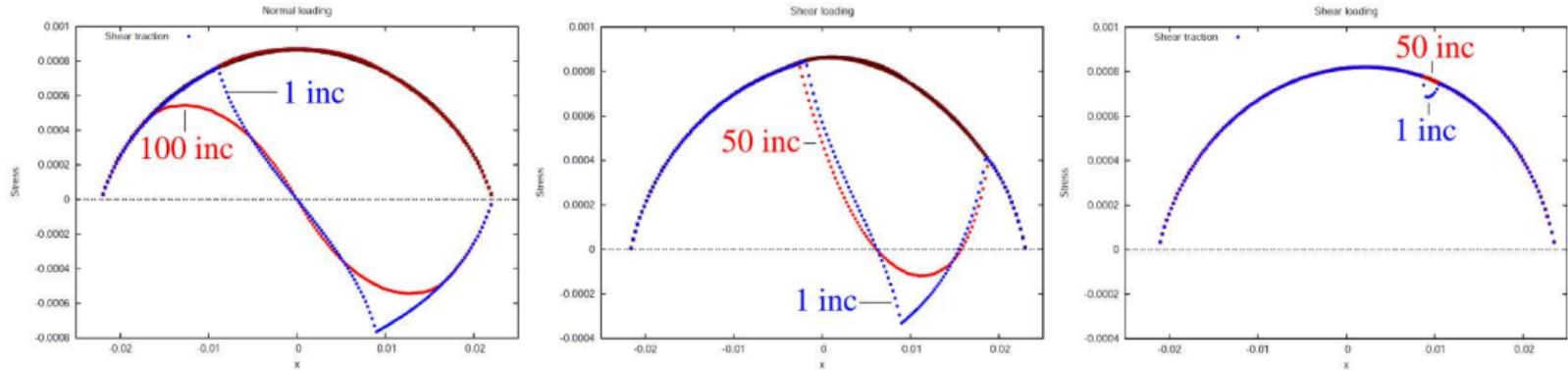
sig12 at maximal tangential load



Shift in 100 increments,  $u_z \sim t$

# Cylinder-plane frictional contact

- Non-conservative problem, history of loading is crucial

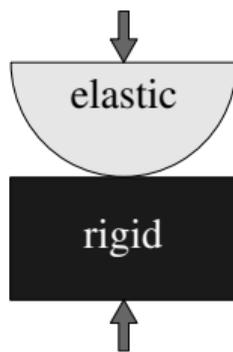


Comparison with: press in 1 increment, shift in 2 increments

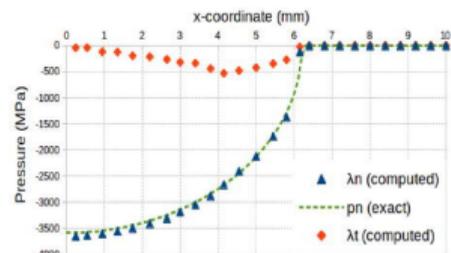
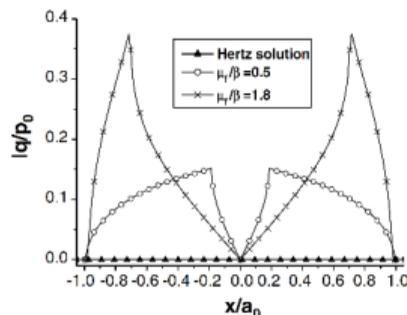
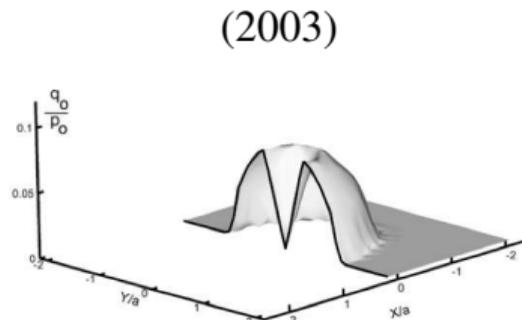
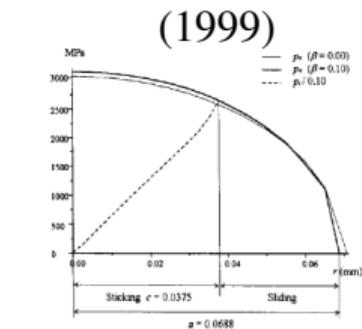
*Before sticking, every point of the contact interface has to pass through the slip zone. It is impossible when loaded too fast.*

# Warning friction!

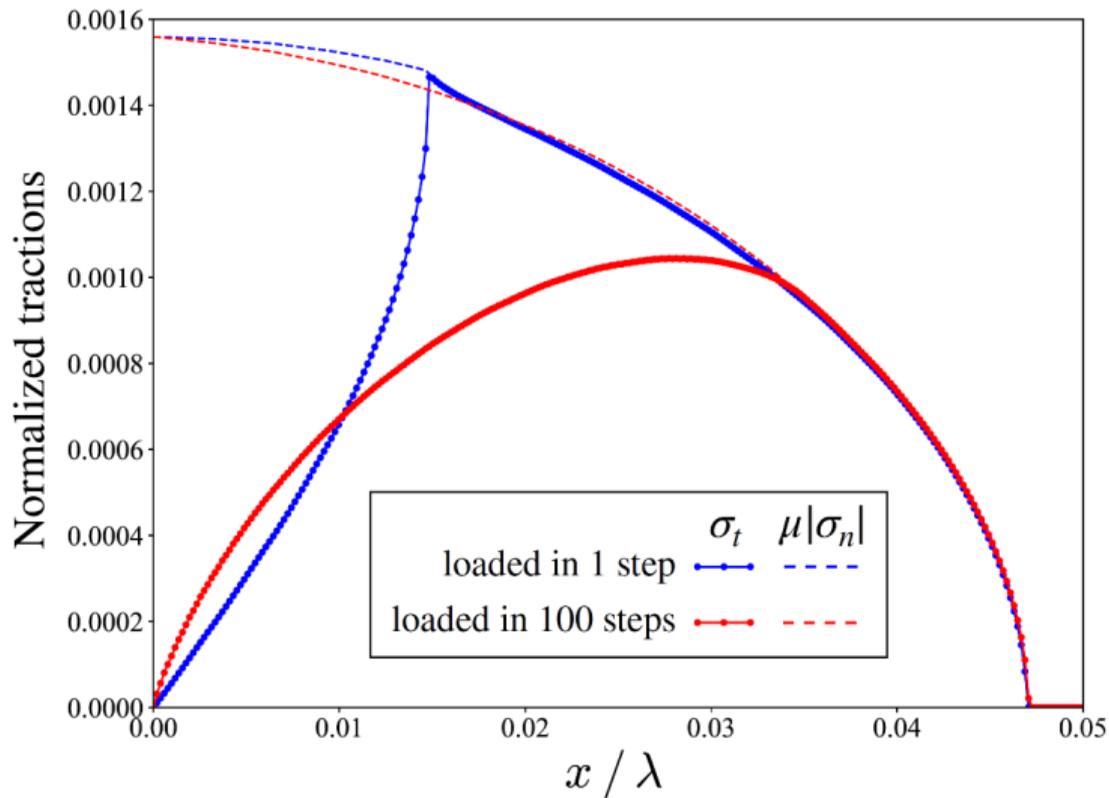
- For dissimilar materials, the *friction matters* even in normal contact
- The problem is thus path-dependent, the B.C. should be changed slowly



Erroneous solutions

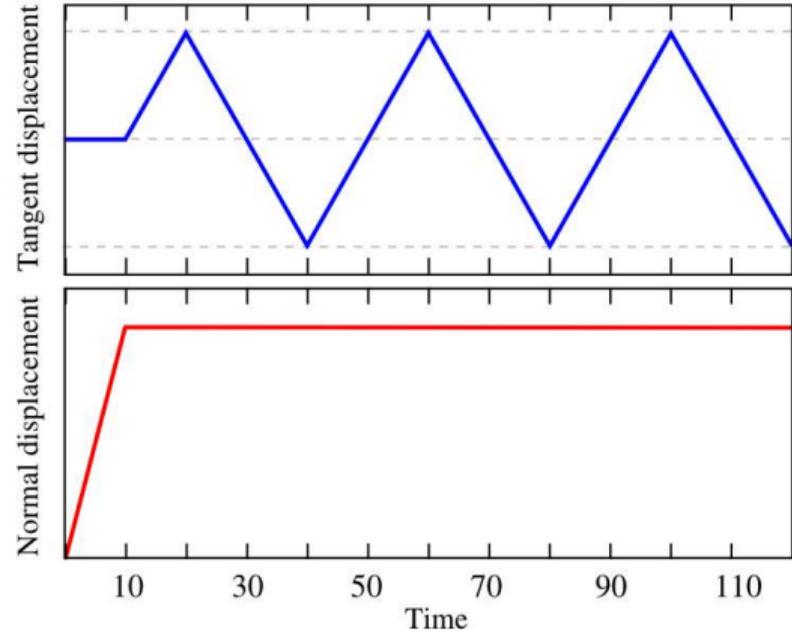
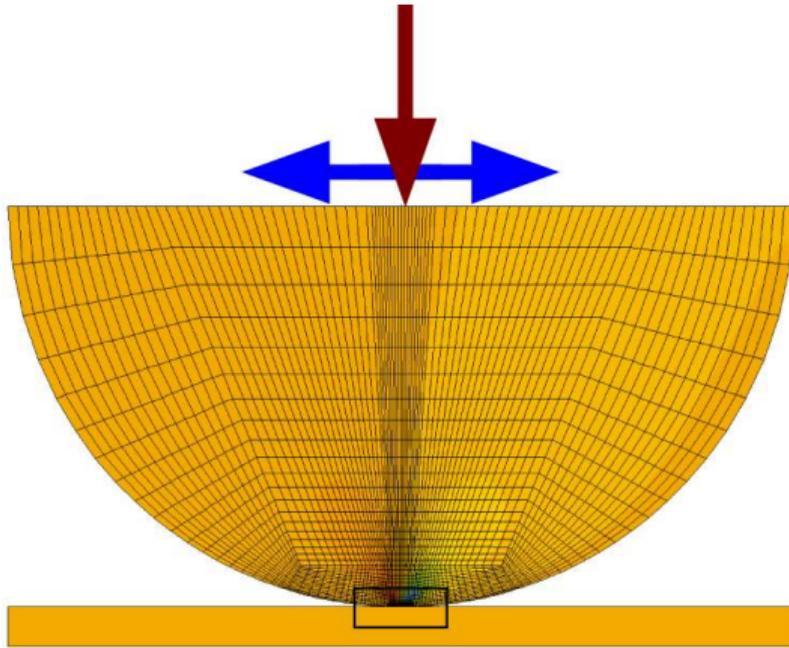


# Warning friction!

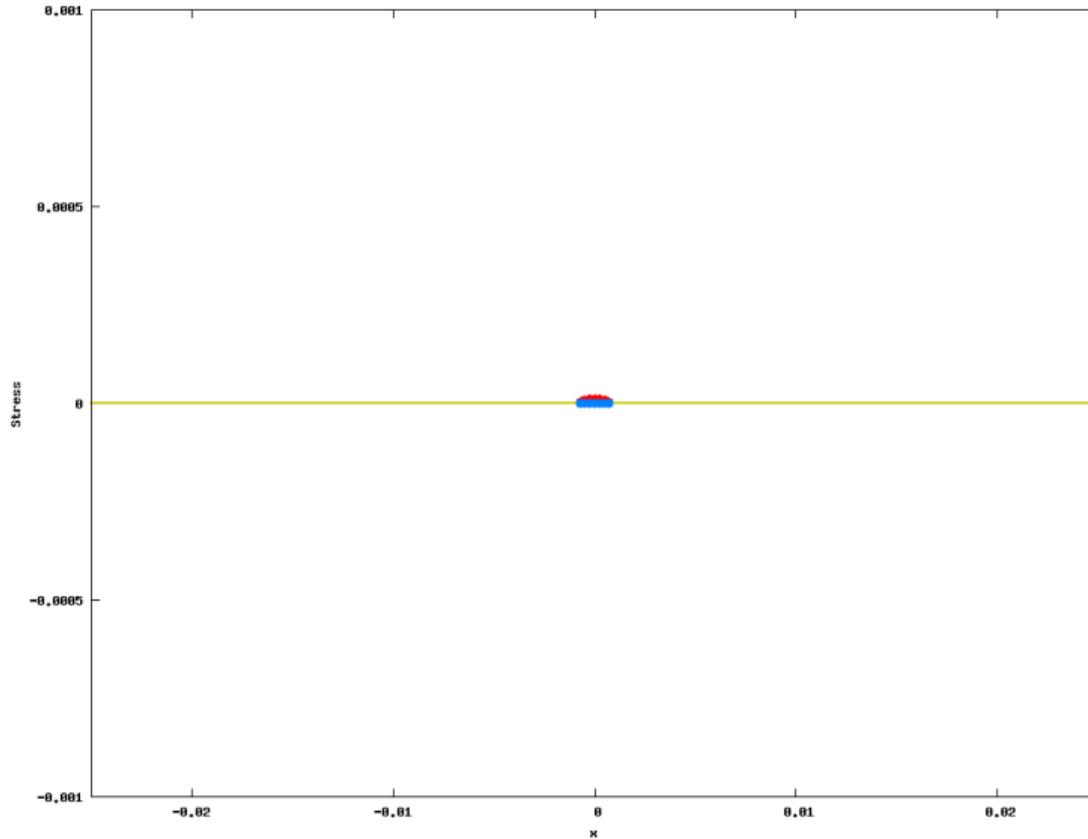


[1] A.G. Shvarts, PhD thesis, MINES ParisTech (2019)

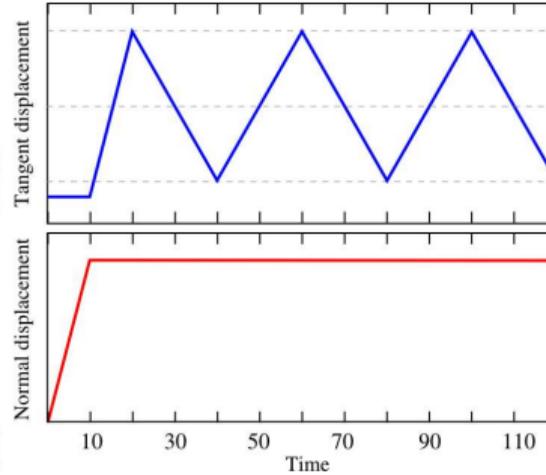
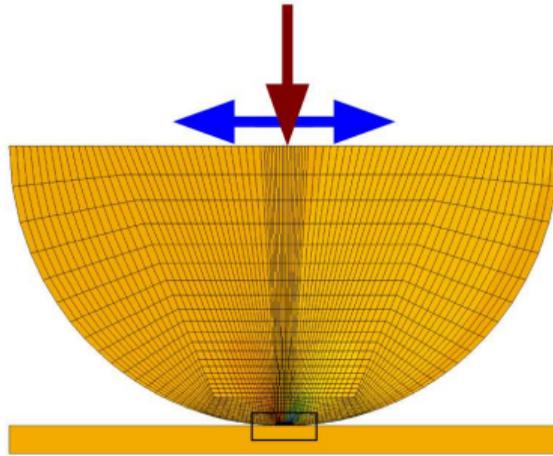
# Sphere-plane frictional contact: cycling



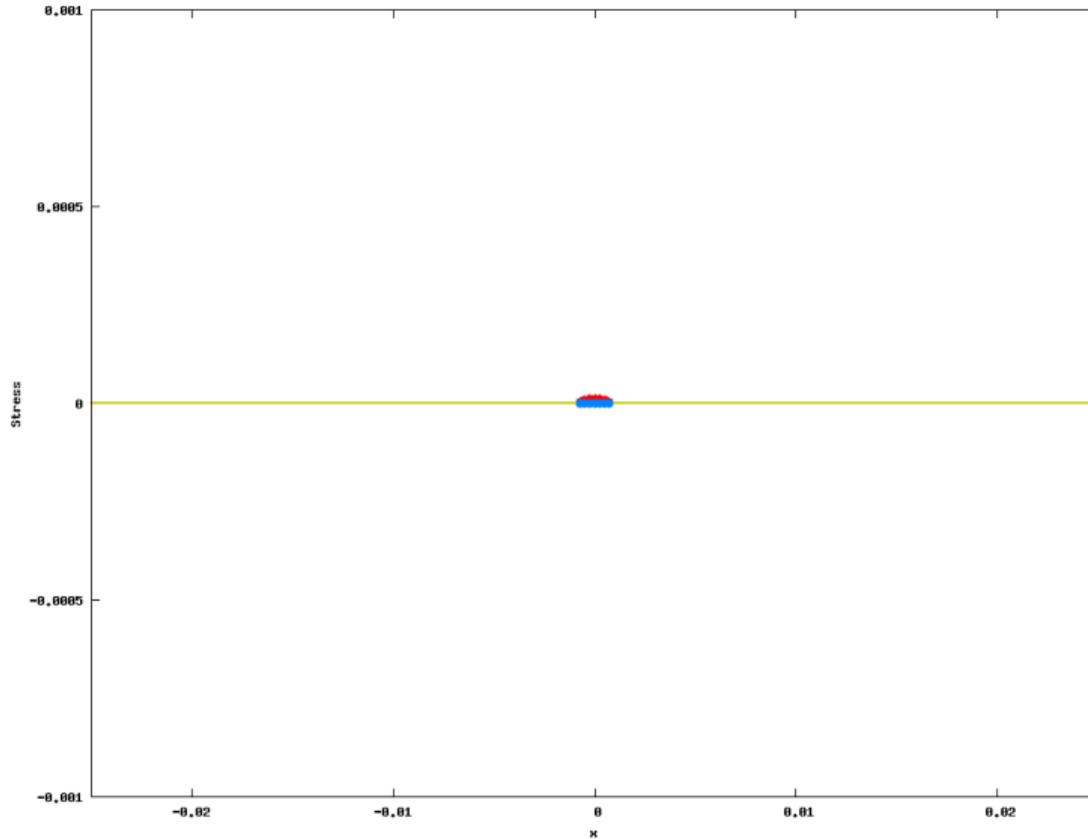
# Sphere-plane frictional contact: cycling



# Sphere-plane frictional contact: cycling

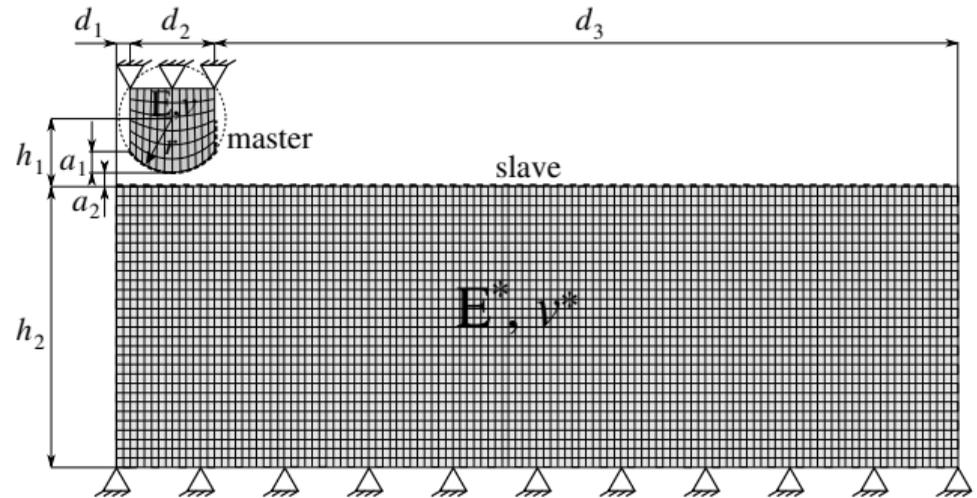


# Sphere-plane frictional contact: cycling



# Shallow ironing test

- Deformable-on-deformable frictional sliding
- Results obtained by different groups<sup>1,2,3,4,5,6</sup> differ significantly
- Local and global friction coefficients may differ



[1] Fischer K. A., Wriggers P., "Mortar based frictional contact formulation for higher order interpolations using the moving friction cone", Computer Methods in Applied Mechanics and Engineering, vol. 195, p. 5020-5036, 2006.

[2] Hartmann S., Oliver J., Cante J. C., Weyler R., Hernández J. A., "A contact domain method for large deformation frictional contact problems. Part 2: Numerical aspects", Computer Methods in Applied Mechanics and Engineering, vol. 198, p. 2607-2631, 2009.

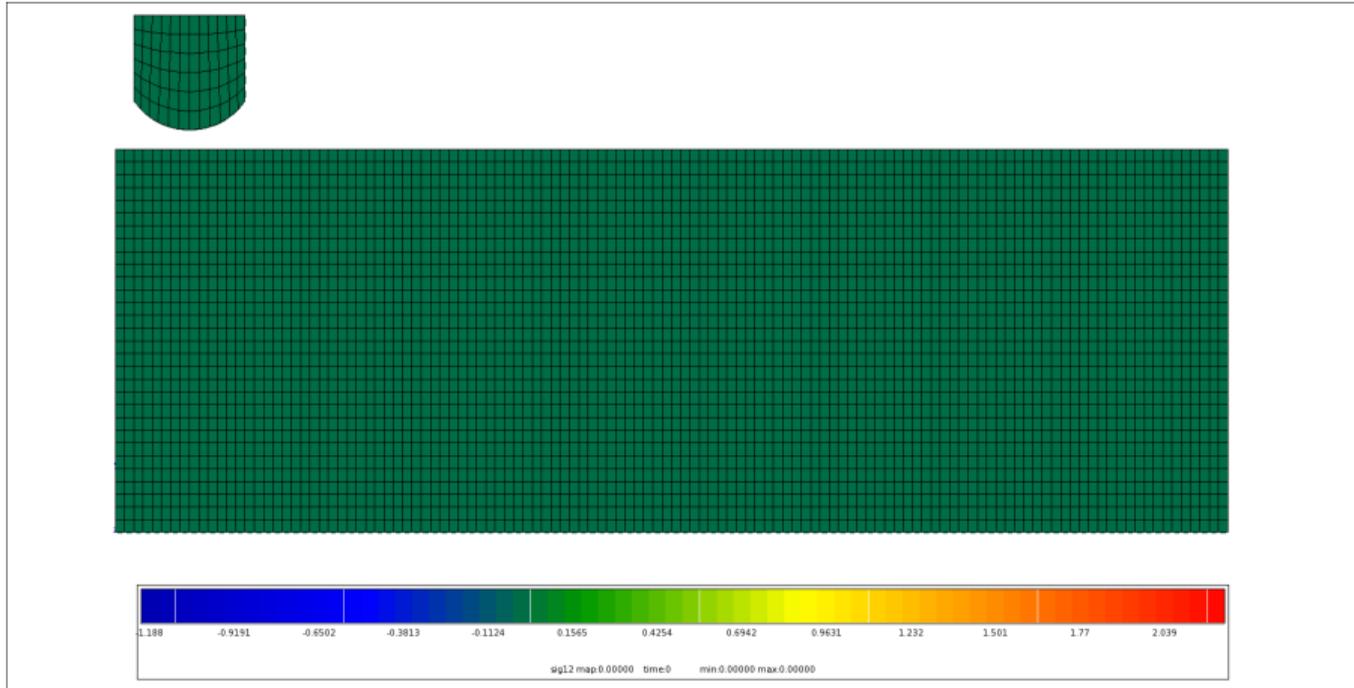
[3] Yastrebov V. A., "Computational contact mechanics: geometry, detection and numerical techniques", Thèse Cdm & Onera, 2011.

[4] Kudawoo A. D., "Problèmes industriels de grande dimension en mécanique numérique du contact : performance, fiabilité et robustesse", Thèse @ LMA & LAMSID, 2012.

[5] Poullos K., Renard Y., "A non-symmetric integral approximation of large sliding frictional contact problems of deformable bodies based on ray-tracing", soumis, 2014.

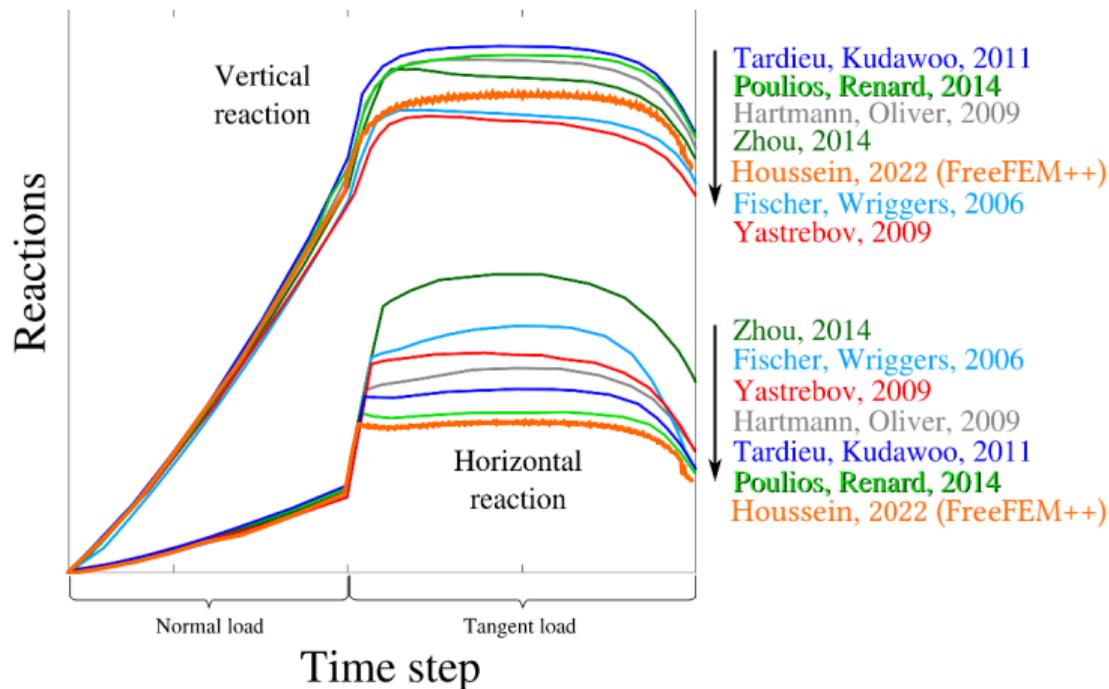
[6] Zhou Lei's blog, <http://kt2008plus.blogspot.de>

# Shallow ironing test



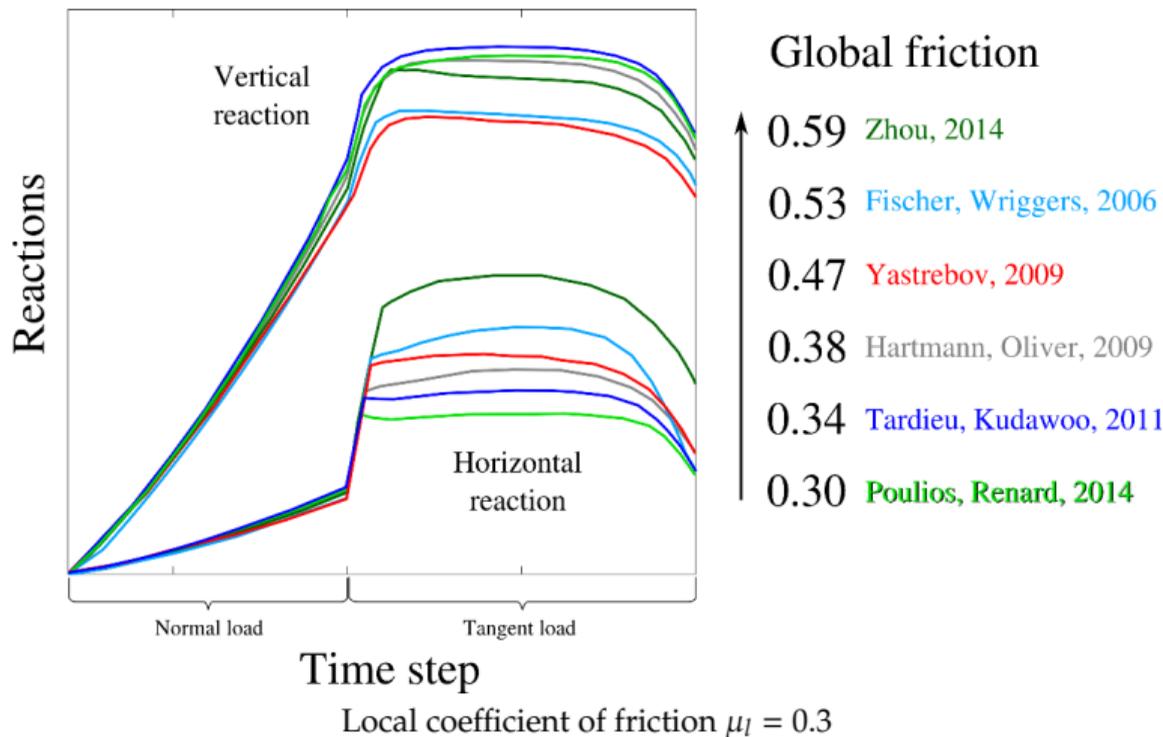
# Shallow ironing test

- No agreement between authors
- Dif. authors used dif. meshes (quadrilateral lin./sq., triangular lin.)
- Dif. authors used either finite or infinitesimal strain formulation



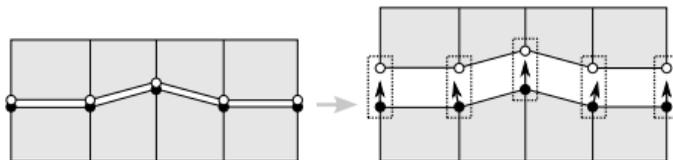
# Shallow ironing test

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- Dif. authors used either finite or infinitesimal strain formulation

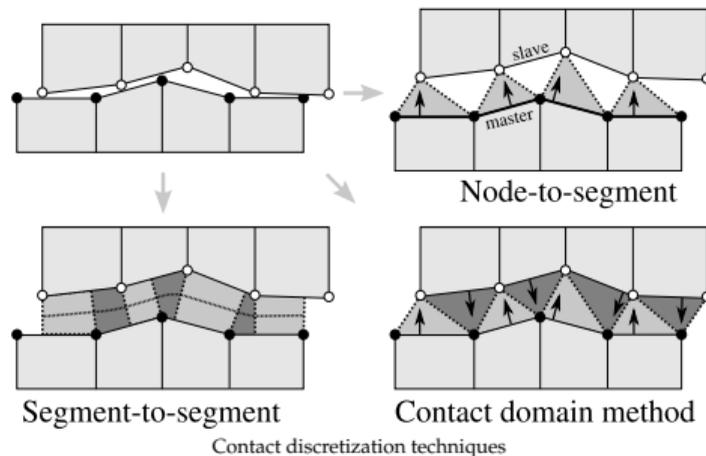


- It's just a tip of the "Computational Contact Mechanics" iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

## Infinitesimal deformation / infinitesimal sliding

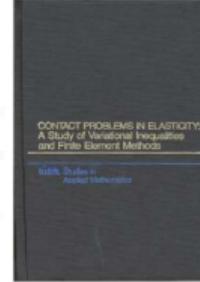


## General case



- It's just a tip of the “Computational Contact Mechanics” iceberg
- Contact discretization and integration
- Smoothing techniques
- Energy conservative methods for dynamics

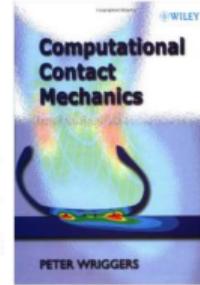
Kikuchi, Oden (1988)



Zhong (1993)



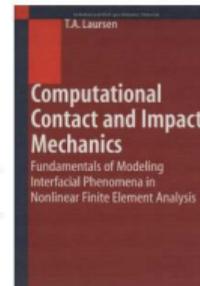
Wriggers (2002)



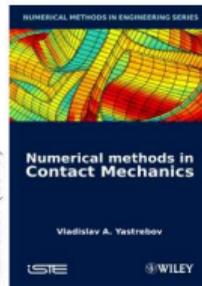
Wriggers, 2<sup>nd</sup> ed. (2006)



Laursen (2002)



Yastrebov (2013)



$\mathcal{L}_a(x, \lambda)$

Thank you for your attention!

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