Flamant's problem

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February 6, 2023

1 Problem statement

In Fig. 1 an elastic half-plane, Young's modulus E and Poisson's ratio ν, is shown. On its surface, a semi-circular groove of radius r is loaded by a distributed pressure $p(\theta) = p_0 \cos(\theta)$.

Problem: Find an induced stress state, deformation state and displacement field. Obtain asymptotic results for $r \to 0$ assuming that the resulting vertical force remains fixed.

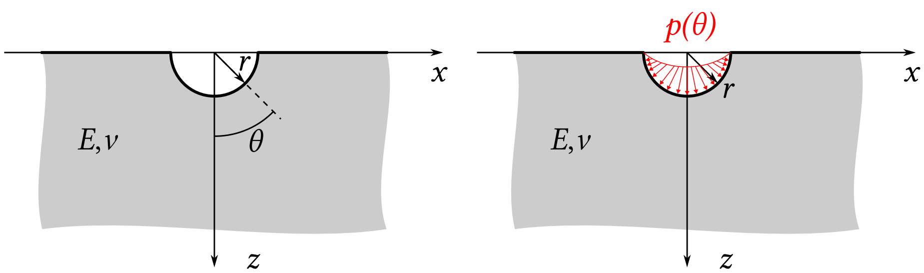

Figure 1: Elastic half-plane with a semi-circular groove on its surface subject to a distributed pressure $p(\theta)$

2 Stress tensor distribution

The stress state is given by the following tensor in polar coordinates

$$\underline{\underline{\sigma}} = -\frac{\alpha \cos(\theta)}{r} \left(\underline{e}_r \otimes \underline{e}_r + \nu \underline{e}_z \otimes \underline{e}_z \right), \quad (1)$$

where $\alpha = r_0 p_0$. The integral of the stress vector over the circular hole gives:

$$-\int_{-\pi/2}^{\pi/2} \underline{\underline{\sigma}} \cdot \underline{e}_r r_0 d\theta = \frac{\alpha\pi}{2} \underline{e}_y = F \underline{e}_y, \quad (2)$$

then

$$\alpha = \frac{2F}{\pi}, \tag{3}$$

where F is the linear density of applied normal force.

3 Strain tensor distribution

The strain tensor is given by

$$\underline{\underline{\varepsilon}} = -\frac{\alpha\cos(\theta)}{rE}\left[(1-\nu^2)\underline{e}_r \otimes \underline{e}_r - \nu(1+\nu)\underline{e}_\theta \otimes \underline{e}_\theta\right] \tag{4}$$

4 Displacement field

The radial displacement can be found by integrating $\varepsilon_{rr} = \partial u_r/\partial r$:

$$u_r = -\frac{\alpha\cos(\theta)(1-\nu^2)}{E}\log(r) + f(\theta), \tag{5}$$

where $f(\theta)$ is an uknown function. The second displacement component u_θ can be found through the expression of $\varepsilon_{\theta\theta} = \frac{1}{r}(\partial u_\theta/\partial\theta + u_r)$, which after integration takes the form:

$$u_\theta = -\frac{\alpha\sin(\theta)\nu(1+\nu)}{E} + \frac{\alpha\sin(\theta)(1-\nu^2)}{E}\log(r) - \int f(\theta)d\theta + g(r), \tag{6}$$

where $g(r)$ is another unknown function. So, we have two unknown functions and will need at least two equations to identify them. The both can be obtained from the fact that $\varepsilon_{r\theta} = 0$, in polar coordinates it has a form:

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right] = 0, \tag{7}$$

or equvalently for non-zero r

$$\frac{\partial u_r}{\partial\theta} - u_\theta + r\frac{\partial u_\theta}{\partial r} = 0. \tag{8}$$

We substitute (5) and (6) in it and obtain:

$$\frac{\partial f(\theta)}{\partial\theta} + \frac{\alpha\sin(\theta)\nu(1+\nu)}{E} + \int f(\theta)d\theta - g(r) - \frac{\alpha\sin(\theta)(1-\nu^2)}{E} + r\frac{\partial g(r)}{\partial r} = 0. \tag{9}$$

After grouping terms that depend solely on r and on θ we obtain the following equality:

$$\frac{\partial f(\theta)}{\partial\theta} + \int f(\theta)d\theta - \frac{\alpha\sin(\theta)(1+\nu)(1-2\nu)}{E} = g(r) - r\frac{\partial g(r)}{\partial r}. \tag{10}$$

Thanks to this separation of variables, both the left and the right hand sides should be equal to the same constant C, and we obtain two equations needed to find $f(\theta)$ and $g(r)$:

$$\begin{cases} \dfrac{\partial f(\theta)}{\partial \theta} + \displaystyle\int f(\theta) d\theta - \dfrac{\alpha \sin(\theta)(1+\nu)(1-2\nu)}{E} = C \\ g(r) - r\dfrac{\partial g(r)}{\partial r} = C \end{cases} \tag{11}$$

We take the derivative of the first and obtain:

$$\frac{\partial^2 f(\theta)}{\partial \theta^2} + f(\theta) = \frac{\alpha \cos(\theta)(1+\nu)(1-2\nu)}{E}. \tag{12}$$

The solution of the homogeneous (for zero right hand part) linear second-order differential equation is given by:

$$f_0(\theta) = A\cos(\theta) + B\sin(\theta), \tag{13}$$

the particular solution we can seek in the form:

$$f_*(\theta) = h(\theta)\sin(\theta), \tag{14}$$

which after its substitution in (12) gives:

$$\frac{\partial^2 h}{\partial \theta^2}\sin(\theta) + 2\frac{\partial h}{\partial \theta}\cos(\theta) = \frac{\alpha\cos(\theta)(1+\nu)(1-2\nu)}{E}, \tag{15}$$

therefore

$$\frac{\partial^2 h}{\partial \theta^2} = 0 \quad \text{and} \quad 2\frac{\partial h}{\partial \theta} = \frac{\alpha(1+\nu)(1-2\nu)}{E}, \tag{16}$$

since we have already $B\sin(\theta)$ in our solution of the homogeneous equation f_0, we keep only the linear term of function $h(\theta) = \alpha(1+\nu)(1-2\nu)\theta/(2E)$:

$$f_*(\theta) = \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta). \tag{17}$$

The full solution for $f(\theta)$ is then given by:

$$\boxed{f(\theta) = A\cos(\theta) + B\sin(\theta) + \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta)}. \tag{18}$$

For the function $g(r)$, from Eq. (11) it immediately follows that

$$g(r) = Er + C. \tag{19}$$

Finally, the displacements are given by:

$$u_r = -\frac{\alpha\cos(\theta)(1-\nu^2)}{E}\log(r) + \underbrace{A\cos(\theta) + B\sin(\theta)}_{\text{Rigid body displacement}} + \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\sin(\theta) \tag{20}$$

$$u_\theta = -\frac{\alpha \sin(\theta)\nu(1+\nu)}{E} + \frac{\alpha \sin(\theta)(1-\nu^2)}{E}\log(r) \underbrace{-A\sin(\theta) + B\cos(\theta)}_{\text{Rigid body displacement}} - \frac{\alpha(1+\nu)(1-2\nu)}{2E}\sin(\theta) + $$
$$+ \frac{\alpha(1+\nu)(1-2\nu)}{2E}\theta\cos(\theta) + \underbrace{Er}_{\text{Rigid body rotation}} + C \tag{21}$$

If we remove rigid body motion, we obtain the following displacements on the surface:

$$\boxed{u_x = -\frac{F(1+\nu)(1-2\nu)}{2E}\text{sign}(x)} \tag{22}$$

$$\boxed{u_y = \frac{2F(1-\nu^2)}{\pi E}\log(|x|) + C} \tag{23}$$

Note that $u_x = u_r \underline{e}_r \cdot \underline{e}_x$ for $\theta = \pm\pi/2$, and $u_y = u_\theta \underline{e}_\theta \cdot \underline{e}_y$ for $\theta = \pm\pi/2$. We also used the expression for α from Eq. (3).