Derivation of the Reynolds Equations from the Navier-Stokes Equations

Vladislav A. Yastrebov

Mines Paris, PSL University, Centre des Matériaux, CNRS UMR 7633, Evry, France

27 février 2025

1 Normalization of the Navier-Stokes Equations

The Navier-Stokes equations for an incompressible fluid ($\nabla \cdot \boldsymbol{v} = 0$) are written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot (\nabla \boldsymbol{v}) + \frac{1}{\rho}\nabla p = \frac{\mu}{\rho}\Delta \boldsymbol{v} + \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity vector field, ρ is the constant and uniform density, μ is the fluid viscosity (Pa·s), and $\boldsymbol{f}$ is the body force density. By expressing $\boldsymbol{v} = u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z$, assuming that body forces arise from gravity with gravitational acceleration g in an arbitrary direction, and rewriting these equations component by component, we obtain :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & = \dfrac{\mu}{\rho}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) + g\alpha_x, \\ \dfrac{\partial v}{\partial t} + \left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) + \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & = \dfrac{\mu}{\rho}\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) + g\alpha_y, \\ \dfrac{\partial w}{\partial t} + \left(u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) + \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & = \dfrac{\mu}{\rho}\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) + g\alpha_z, \end{cases} \tag{3}$$

with coefficients $\alpha_i \sim 1$ such that $\alpha_x^2 + \alpha_y^2 + \alpha_z^2 = 1$, determining the projection of gravitational acceleration onto the OX, OY, OZ axes. These equations can be simplified in the case where the fluid flow occurs in a thin layer of characteristic thickness h_0 (in the OZ direction) and in the XY plane over a characteristic distance L with $h_0 \ll L$. Let us begin by introducing the dimensionless variables for coordinates and velocities :

$$\begin{aligned} x' &= \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{h_0}, \\ u' &= \frac{u}{U_0}, \quad v' = \frac{v}{U_0}, \quad w' = \frac{w}{V_0}, \end{aligned} \tag{4}$$

where U_0 and V_0 are characteristic velocities in the XY plane and in the V_0 direction, respectively. Logically, the characteristic velocity in the flow plane U_0 should be greater than in the thickness direction of the layer V_0. To demonstrate this, we rewrite equation (2) using normalized variables, obtaining :

$$\frac{U_0}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{V_0}{h_0}\frac{\partial w'}{\partial z'} = 0. \tag{5}$$

The derivatives of the normalized velocities with respect to the normalized coordinates vary in the same way, thus giving equal weight to all components, it is necessary to demand that $U_0/L \sim V_0/h_0$; we can therefore choose :

$$V_0 = U_0\frac{h_0}{L}. \tag{6}$$

The characteristic time is associated with this ratio :

$$t_0 = \frac{L}{U_0} \tag{7}$$

The pressure p and the time t will also be normalized :

$$p' = \frac{p}{p_0}, \quad t' = \frac{t}{t_0}. \tag{8}$$

By substituting all variables into equations (3) we obtain :

$$\begin{cases} \frac{U_0}{t_0}\frac{\partial u'}{\partial t'} + \frac{U_0^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) + \frac{p_0}{\rho L}\frac{\partial p'}{\partial x'} = \frac{\mu U_0}{\rho L^2}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) + \frac{\mu U_0}{\rho h_0^2}\frac{\partial^2 u'}{\partial z'^2} + g\alpha_x, \\ \frac{U_0}{t_0}\frac{\partial v'}{\partial t'} + \frac{U_0^2}{L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right) + \frac{p_0}{\rho L}\frac{\partial p'}{\partial y'} = \frac{\mu U_0}{\rho L^2}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) + \frac{\mu U_0}{\rho h_0^2}\frac{\partial^2 v'}{\partial z'^2} + g\alpha_y, \\ \frac{U_0 h_0}{t_0 L}\frac{\partial w'}{\partial t'} + \frac{U_0^2 h_0}{L^2}\left(u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'}\right) + \frac{p_0}{h_0 \rho}\frac{\partial p'}{\partial z'} = \frac{\mu U_0 h_0}{L^3 \rho}\left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2}\right) + \frac{\mu U_0}{L h_0 \rho}\frac{\partial^2 w'}{\partial z'^2} + g\alpha_z, \end{cases} \tag{9}$$

Equation in the "plane" : By multiplying the first two equations of (9) by $\rho h_0^2/(\mu U_0)$, we obtain :

$$\begin{cases} \frac{\rho h_0^2}{\mu t_0}\frac{\partial u'}{\partial t'} + \frac{\rho h_0^2 U_0}{\mu L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) + \boxed{\frac{h_0^2 p_0}{\mu U_0 L}}\frac{\partial p'}{\partial x'} = \frac{h_0^2}{L^2}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) + \frac{\partial^2 u'}{\partial z'^2} + \frac{\rho h_0^2 g}{\mu U_0}\alpha_x, \\ \frac{\rho h_0^2}{\mu t_0}\frac{\partial v'}{\partial t'} + \frac{\rho h_0^2 U_0}{\mu L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right) + \boxed{\frac{h_0^2 p_0}{\mu U_0 L}}\frac{\partial p'}{\partial y'} = \frac{h_0^2}{L^2}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) + \frac{\partial^2 v'}{\partial z'^2} + \frac{\rho h_0^2 g}{\mu U_0}\alpha_y \end{cases} \tag{10}$$

To retain the term containing the pressure gradient, the reference pressure should be chosen such that $h_0^2 p_0 \sim \mu U_0 L$, i.e.,

$$p_0 = \frac{\mu U_0 L}{h_0^2}. \tag{11}$$

At the same time, we replace t_0 with L/U_0 and introduce the small parameter $\epsilon = h_0/L \ll 1$, obtaining :

$$\begin{cases} \epsilon^2 \boxed{\frac{\rho U_0 L}{\mu}}\left(\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) + \frac{\partial p'}{\partial x'} = \epsilon^2\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) + \frac{\partial^2 u'}{\partial z'^2} + \epsilon^2 \frac{\rho U_0 L}{\mu}\boxed{\frac{Lg}{U_0^2}}\alpha_x, \\ \epsilon^2 \boxed{\frac{\rho U_0 L}{\mu}}\left(\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right) + \frac{\partial p'}{\partial y'} = \epsilon^2\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) + \frac{\partial^2 v'}{\partial z'^2} + \epsilon^2 \frac{\rho U_0 L}{\mu}\boxed{\frac{Lg}{U_0^2}}\alpha_y, \end{cases} \tag{12}$$

The dimensionless factor preceding the acceleration is called the *Reynolds number* :

$$\mathrm{Re} = \frac{\rho U_0 L}{\mu}, \tag{13}$$

which represents the ratio of inertia to viscosity.

The two factors in front of the body force direction include the Reynolds number and another dimensionless number called the *Froude number*, given by :

$$\mathrm{Fr} = \frac{U_0^2}{Lg} \tag{14}$$

which characterizes the ratio between kinetic energy and potential energy due to gravity.

Taking all these numbers into account, we obtain the final equation in the XY plane :

$$\begin{cases} \epsilon^2 \mathrm{Re}\left(\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) + \frac{\partial p'}{\partial x'} = \epsilon^2\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) + \frac{\partial^2 u'}{\partial z'^2} + \epsilon^2 \frac{\mathrm{Re}}{\mathrm{Fr}}\alpha_x, \\ \epsilon^2 \mathrm{Re}\left(\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right) + \frac{\partial p'}{\partial y'} = \epsilon^2\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right) + \frac{\partial^2 v'}{\partial z'^2} + \epsilon^2 \frac{\mathrm{Re}}{\mathrm{Fr}}\alpha_y, \end{cases} \tag{15}$$

For the case where the Reynolds number remains small such that $\epsilon^2\mathrm{Re} \ll 1$, and if the gravitational contribution remains of order one, $1/\mathrm{Fr} \sim 1$ (since this term is multiplied by $\epsilon^2\mathrm{Re}$), then for $\epsilon \to 0$, the equations (15) reduce to terms that do not contain ϵ :

$$\begin{cases} \dfrac{\partial p'}{\partial x'} &= \dfrac{\partial^2 u'}{\partial z'^2}, \\ \dfrac{\partial p'}{\partial y'} &= \dfrac{\partial^2 v'}{\partial z'^2}, \end{cases} \tag{16}$$

Equation in the thickness : Taking into account all the normalizations mentioned above, the third equation of (9) becomes :

$$\frac{U_0^2 h_0}{L^2}\frac{\partial w'}{\partial t'}+\frac{U_0^2 h_0}{L^2}\left(u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}\right)+\frac{\mu U_0 L}{h_0^3\rho}\frac{\partial p'}{\partial z'}=\frac{\mu U_0 h_0}{L^3\rho}\left(\frac{\partial^2 w'}{\partial x'^2}+\frac{\partial^2 w'}{\partial y'^2}\right)+\frac{\mu U_0}{L h_0 \rho}\frac{\partial^2 w'}{\partial z'^2}+g\alpha_z \tag{17}$$

Focusing on the pressure gradient $\partial p'/\partial z'$, we multiply everything by $h_0^3\rho/(\mu U_0 L)$, obtaining the final equation in the thickness direction OZ :

$$\epsilon^4\mathrm{Re}\left(\frac{\partial w}{\partial t}+u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}\right)+\frac{\partial p'}{\partial z'}=\epsilon^4\left(\frac{\partial^2 w'}{\partial x'^2}+\frac{\partial^2 w'}{\partial y'^2}\right)+\epsilon^2\frac{\partial^2 w'}{\partial z'^2}+\epsilon^3\frac{\mathrm{Re}}{\mathrm{Fr}}\alpha_z \tag{18}$$

It can be observed that, under the previously introduced assumptions that $\epsilon^2\mathrm{Re} \ll 1$ and $1/\mathrm{Fr} \sim 1$ or larger, there is only one term that is not multiplied by the small parameter ϵ :

$$\frac{\partial p'}{\partial z'} = 0. \tag{19}$$

It follows that the pressure remains uniform in the thickness. However, if the term related to gravity remains significant, the pressure will be an affine function of the coordinate z :

$$\frac{\partial p'}{\partial z'} = \epsilon^3\frac{\mathrm{Re}}{\mathrm{Fr}}\alpha_z \quad \Rightarrow \quad p' = \epsilon^3\alpha_z\frac{\mathrm{Re}}{\mathrm{Fr}}z' \quad \Leftrightarrow \quad p = p_0 + \rho g \alpha_z z \tag{20}$$

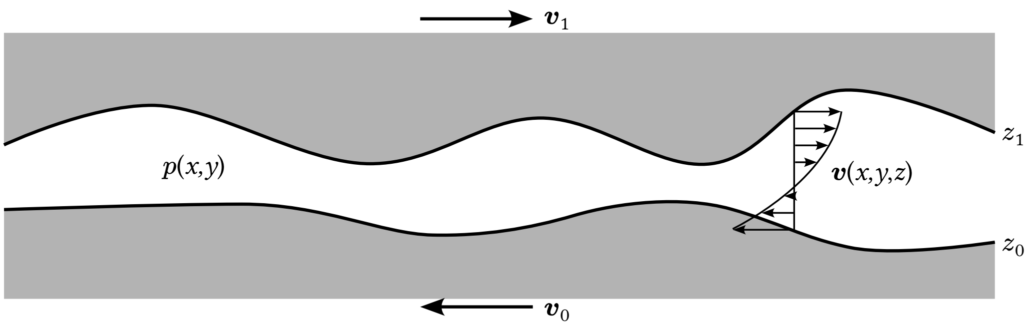

FIGURE 1 – Fluid flow between two walls.

2 Reynolds Equation for Thin-Film Flow

Since the pressure does not depend on the coordinate z (cf. (19)), equations (21) are easily integrated to show that :

$$\begin{cases} u' = \dfrac{z'^2}{2}\dfrac{\partial p'}{\partial x'} + f_u(x',y',t')z' + g_u(x',y',t'), \\ v' = \dfrac{z'^2}{2}\dfrac{\partial p'}{\partial y'} + f_v(x',y',t')z' + g_v(x',y',t'), \end{cases} \tag{21}$$

For thin-film flow bounded by two surfaces $z_0(x,y) \leq z \leq z_1(x,y)$ (cf. Fig. 1), the boundary conditions that determine the functions f_u, g_u, f_v, g_v may vary :

1. Free surface (absence of shear stress) at $z' = z_0'(x',y')$ or $z' = z_1'(x',y')$: $\dfrac{\partial u'}{\partial z'} = \dfrac{\partial v'}{\partial z'} = 0$

2. Adherence to a wall at $z' = z'_0(x', y')$ or $z' = z'_1(x', y')$: $u'(z'_0, z'_1) = U'$ and $v'(z'_0, z'_1) = V'$, where $\{U', V'\}$ are the components of the normalized velocity of the corresponding wall.

Regardless of the boundary conditions, it can be observed in (21) that the velocity in the thickness direction is described by a second-order polynomial. Note that this remains true even if the last term containing the Froude number Fr is not neglected.

Let us consider the case of flow between two rigid walls $z'_0 = 0$ and $z'_1 = h'(x, y, t) > 0$, where $h' = h/h_0$ and $h(x, y)$ is the film thickness. The walls are thus separated on average by h_0 and move at velocities $\boldsymbol{v}'_0 = u'_0\boldsymbol{e}_x + v'_0\boldsymbol{e}_y$ and $\boldsymbol{v}'_1 = u'_1\boldsymbol{e}_x + v'_1\boldsymbol{e}_y$, respectively.

Formulated in this way, the resulting equations are used to describe lubrication, flow in fractured media (rocks), and sealing systems.

From equation (21) with pure adhesion conditions, we obtain $u'(0) = u'_0$, so $g_u = u'_0$, and $v'(0) = v'_0$, so $g_v = v'_0$.

For $u'(h') = u'_1$ and $v'(h') = v'_1$, we obtain $f_u = (u'_1 - u'_0)/h' - 0.5h'\partial p'/\partial x'$ and $f_v = (v'_1 - v'_0)/h' - 0.5h'\partial p'/\partial y'$.

Finally, the fluid velocity between the two walls is given by :

$$\boldsymbol{v}'(z) = \frac{z'}{2}(z' - h')\nabla' p' + \frac{z'}{h'}(\boldsymbol{v}'_1 - \boldsymbol{v}'_0) + \boldsymbol{v}_0, \tag{22}$$

where $\nabla' = \partial/\partial x'\boldsymbol{e}_x + \partial/\partial y'\boldsymbol{e}_y$. Integrating this velocity over the thickness, we obtain the components of the flux $\boldsymbol{q}' = q'_x\boldsymbol{e}_x + q'_y\boldsymbol{e}_y$:

$$\begin{aligned} q'_x &= \int_{z'_0}^{z'_1} u(z')dz = -\frac{h'^3}{12}\frac{\partial p'}{\partial x'} + \frac{h'}{2}(u'_1 + u'_0), \\ q'_y &= \int_{z'_0}^{z'_1} v(z')dz = -\frac{h'^3}{12}\frac{\partial p'}{\partial y'} + \frac{h'}{2}(v'_1 + v'_0) \end{aligned} \tag{23}$$

where we used the fact that $z'_0 = 0$ and $z'_1 = h'$. Thus, the flux vector is given by :

$$\boldsymbol{q} = -\frac{h'^3}{12}\nabla' p' + \frac{h'}{2}(\boldsymbol{v}_0 + \boldsymbol{v}_1) \tag{24}$$

Mass conservation implies that the divergence of the flux $\nabla \cdot \boldsymbol{q}$ must be balanced by the change in volume (height) $\partial h/\partial t$, i.e.,

$$\frac{\partial h'}{\partial t'} + \nabla' \cdot \boldsymbol{q}' = 0. \tag{25}$$

To explicitly express the first term, we start from the time derivative of the height :

$$\frac{dh'}{dt'} = w'_1 - w'_0 = \frac{\partial h'}{\partial t'} + (u'_1 + u'_0)\frac{\partial h'}{\partial x'} + (v'_1 + v'_0)\frac{\partial h'}{\partial y'}, \tag{26}$$

where $w'_1 - w'_0$ is the relative velocity in the thickness direction between the two walls. Thus, we obtain the partial time derivative :

$$\frac{\partial h'}{\partial t'} = w'_1 - w'_0 - (\boldsymbol{v}'_1 + \boldsymbol{v}'_0) \cdot \nabla' h \tag{27}$$

Substituting equations (23) and (27) into (25), we obtain the normalized Reynolds equation for an incompressible fluid :

$$\boxed{\nabla' \cdot \left(\frac{h'^3}{12}\nabla' p'\right) = w'_1 - w'_0 - \frac{1}{2}(\boldsymbol{v}'_1 + \boldsymbol{v}'_0) \cdot \nabla' h'.} \tag{28}$$

Note that this is an equation for a scalar pressure field to be determined in the "flow" plane, $p'(x', y')$. The velocity field is fully defined by the pressure gradient and the velocity of the walls (22). By removing the normalization : $\nabla' = L\nabla$, $h' = h/h_0$, $\boldsymbol{v}' = \boldsymbol{v}/U_0$, $w' = wL/(U_0h_0)$, $p' = ph_0^2/(\mu U_0 L)$, we obtain the classical Reynolds equation for an incompressible fluid with viscosity μ :

$$\boxed{\nabla \cdot \left(\frac{h^3}{12\mu}\nabla p\right) = w_1 - w_0 - \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_0) \cdot \nabla h.} \tag{29}$$