Practical Work: Integration of Boussinesq and Cerutti solutions for frictional contact of a rigid sphere on an elastic half-space

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1. Introduction

In this practical work we will work with the integration of Boussinesq and Cerutti solutions (see Fig. 1). The Boussinesq solution makes a link between concentrated normal force N and displacements/stresses developped in the isotropic elastic half-space. The Cerutti solution provides the same result for the concentrated tangential force T on a half-space. Therefore, assuming linearity of the problem, we can use the principle of superposition to obtain the solution for a general case of arbitrary distributed tranctions on the surface of the half-space by integrating the Boussinesq and Cerutti solutions. We consider the following elastic properties of the half-space: Young modulus E, Poisson ratio ν and shear modulus $G = E/(2(1+\nu))$.

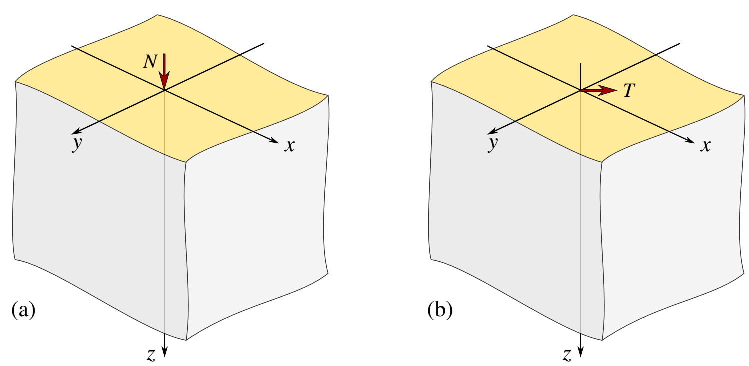

Figure 1: Schematic of the problem: (a) Boussinesq problem, (b) Cerutti problem.

2. Boussinesq solution

The Boussinesq solution for a concentrated normal force F applied at $\{x', y'\}$ is given by the following expressions for the stresses at point $\{x, y, z\}$:

$$
\begin{aligned}
\rho &= \sqrt{(x-x')^2+(y-y')^2+z^2}\\
r &= \sqrt{(x-x')^2+(y-y')^2}\\
\sigma_x^B &= \frac{N}{2\pi}\left(\frac{1-2\nu}{r^2}\left(\frac{1-z/\rho}{r^2}\left((x-x')^2-(y-y')^2\right)+\frac{z(y-y')^2}{\rho^3}\right)-\frac{3z(x-x')^2}{\rho^5}\right)\\
\sigma_y^B &= \frac{N}{2\pi}\left(\frac{1-2\nu}{r^2}\left(\frac{1-z/\rho}{r^2}\left((y-y')^2-(x-x')^2\right)+\frac{z(x-x')^2}{\rho^3}\right)-\frac{3z(y-y')^2}{\rho^5}\right)\\
\sigma_z^B &= -3\frac{N}{2\pi}\frac{z^3}{\rho^5}\\
\sigma_{xy}^B &= \frac{N}{2\pi}(x-x')(y-y')\left(\frac{1-2\nu}{r^2}\left(\frac{1-z/\rho}{r^2}-\frac{z}{\rho^3}\right)-\frac{3z}{\rho^5}\right)\\
\sigma_{xz}^B &= -3\frac{N}{2\pi}\frac{z^2(x-x')}{\rho^5}\\
\sigma_{yz}^B &= -3\frac{N}{2\pi}\frac{z^2(y-y')}{\rho^5}.
\end{aligned}
$$

Associated displacements on the surface of the half-space at $\{x, y\}$ are given by:

$$
\begin{aligned}
u_x^B &= \frac{N(1-2\nu)}{4\pi G}\frac{x-x'}{r^2}\\
u_y^B &= \frac{N(1-2\nu)}{4\pi G}\frac{y-y'}{r^2}\\
u_z^B &= \frac{N(1-\nu)}{2\pi G}\frac{1}{r} = \frac{F(1-\nu^2)}{\pi E r},
\end{aligned}
$$

note that the vertical displacement is of positive sign since the axis OZ is oriented downwards in the half-space.

These expressions could be reformulated as follows:

$$\boxed{\sigma_{ij}^B(x,y,z) = N S_{ij}^B(x-x', y-y', z, \nu)}$$

$$\boxed{u_i^B(x,y) = N D_i^B(x-x', y-y', G, \nu)}$$

where $S_{ij}^B(\Delta x, \Delta y, z, \nu)$ and $D_i^B(\Delta x, \Delta y, G, \nu)$ are the Boussinesq kernels for stresses and displacements, respectively. Note that even though we included the Poisson ratio ν in the expressions for the Boussinesq kernels, $\sigma_z, \sigma_{xz}, \sigma_{yz}$ components are independent of ν.

2. Cerutti solution

The Cerutti solution for a concentrated tangential force T_x applied at $\{x', y'\}$ is given by the following expressions for the stresses at point $\{x, y, z\}$:

$$
\begin{aligned}
\sigma_x^C &= \frac{T_x}{2\pi}\left(-\frac{3(x-x')^3}{\rho^5}+\right.\\
&\left.+(1-2\nu)\left(\frac{x-x'}{\rho^3}-\frac{3(x-x')}{\rho(\rho+z)^2}+\frac{(x-x')^3}{\rho^3(\rho+z)^2}+\frac{2(x-x')^3}{\rho^2(\rho+z)^3}\right)\right)\\
\sigma_y^C &= \frac{T_x}{2\pi}\left(-\frac{3(x-x')(y-y')^2}{\rho^5}+\right.\\
&\left.+(1-2\nu)\left(\frac{x-x'}{\rho^3}-\frac{x-x'}{\rho(\rho+z)^2}+\frac{(x-x')(y-y')^2}{\rho^3(\rho+z)^2}+\frac{2(x-x')(y-y')^2}{\rho^2(\rho+z)^3}\right)\right)\\
\sigma_z^C &= -\frac{3T_x}{2\pi}\frac{(x-x')z^2}{\rho^5}\\
\sigma_{xy}^C &= \frac{T_x}{2\pi}\left(-\frac{3(x-x')^2(y-y')}{\rho^5}+\right.\\
&\left.+(1-2\nu)\left(-\frac{(y-y')}{\rho(\rho+z)^2}+\frac{(x-x')^2(y-y')}{\rho^3(\rho+z)^2}+\frac{2(x-x')^2(y-y')}{\rho^2(\rho+z)^3}\right)\right)\\
\sigma_{xz}^C &= -\frac{3T_x}{2\pi}\frac{(x-x')(y-y')z}{\rho^5}\\
\sigma_{yz}^C &= -\frac{3T_x}{2\pi}\frac{(x-x')^2z}{\rho^5}
\end{aligned}
$$

Associated displacements on the surface of the half-space at $\{x, y\}$ are given by:

$$
\begin{aligned}
u_x^C &= \frac{T_x}{4\pi G}\left(\frac{1}{r}-\frac{(x-x')^2}{r^3}+(1-2\nu)\left(\frac{1}{r}-\frac{(x-x')^2}{r^3}\right)\right)\\
u_y^C &= -\frac{T_x}{4\pi G}\cdot\frac{(x-x')(y-y')2\nu}{r^3}\\
u_z^C &= \frac{T_x}{4\pi G}\cdot\frac{(1-2\nu)(x-x')}{r^2}.
\end{aligned}
$$

By analogy with the Boussinesq solution, these expressions could be reformulated as follows:

$$\boxed{\sigma_{ij}^C(x,y,z) = T_x S_{ij}^C(x-x', y-y', z, \nu)}$$

$$\boxed{u_i^C(x,y) = T_x D_i^C(x-x', y-y', G, \nu)}$$

where $S_{ij}^C(\Delta x, \Delta y, z, \nu)$ and $D_i^C(\Delta x, \Delta y, G, \nu)$ are the Cerutti kernels for stresses and displacements, respectively. Note that even though we included the Poisson ratio ν in the expressions for the Boussinesq kernels, $\sigma_z, \sigma_{xz}, \sigma_{yz}$ components are independent of ν.

3. Integration of Boussinesq and Cerutti solutions

Consider a pressure distribution $p(x, y)$ and tangential OX tractions $\boldsymbol{q}(x, y) = q(x, y)\boldsymbol{e}_x$ applied on region Ω of the surface of the half-space. Assuming linear elasticity, we can superpose the Boussinesq and Cerutti solutions to obtain the following expressions for the stresses at point $\{x, y, z\}$:

$$\sigma_{ij}(x, y, z) = \int\limits_{\Omega} \left(S^B_{ij}(x - x', y - y', z, \nu)p(x', y') + S^C_{ij}(x - x', y - y', z, \nu)q(x', y') \right) dx'dy'. \tag{1}$$

For the displacements on the surface, the integrals are the following:

$$u_i(x, y) = \int\limits_{\Omega} \left(D^B_i(x - x', y - y', G, \nu)p(x', y') + D^C_i(x - x', y - y', G, \nu)q(x', y') \right) dx'dy'. \tag{2}$$

4. Problem

Consider now that we have a rigid parabolic indenter coming in contact with a half-space. The induced pressure is then given by Hertzian solution

$$p(x, y) = \begin{cases} p_0\sqrt{1 - r^2/a^2}, & \text{if } r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \le a \\ 0, & \text{elsewhere,} \end{cases}$$

where p_0 is the pressure in the center. Let us now assume that the indenter is in tangential frictional motion in OX direction. Then assuming that normal and tangential loads are decoupled, we can assume that the developed frictional tractions are given by

$$q(x, y) = \begin{cases} p_0\mu\sqrt{1 - r^2/a^2}, & \text{if } r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \le a \\ 0, & \text{elsewhere,} \end{cases}$$

where μ is the coefficient of friction. So, now we know the tractions on the surface and we would like to know what are the induced stresses and displacements.

5. Task

You are provided with `BoussinesqCeruttiSolver.py` Python script, which performs the integration from Eqs. (1,2).

1. Set up appropriate values for indenter's radius and the indentation force.
2. Determine the associated contact pressure and contact radius using Hertz' equations.
3. Introduce relevant data in the provided code and set the friction coefficient to zero. Analyze the stresses in a section passing through the center of the indentation. Analyze induced surface displacements.

4. Increase the friction coefficient. Check how the stresses and displacements vary.
5. Construct a code that plots stress field on the surface of the half-space using the examples provided in the code.
6. Deduce and plot the first principle stress component $\sigma_1(x, y)$ on the surface of the half-space for different coefficients of friction.
7. Check the analytical solution by Hamilton & Goodman (attached paper), compare with your integrated stress field.
8. Quantify the validity of the assumption on the decoupling of normal and tangential tractions on the form of the indenter accommodated by the resulting displacement.