Practical Work: Solving Reynolds equation using finite difference method

Vladislav A. Yastrebov

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1. Introduction

Reynolds equation for tangential sliding between two surfaces or profiles is given by (see Fig. 1):

$$\nabla \cdot \left(h^3 \nabla P\right) = 6\mu \boldsymbol{V}_0 \cdot \nabla h.$$

h is the film thickness and P is the fluid pressure. The viscosity is assumed to be constant $\mu = \text{const}$ which is a valid assumption for hydrodynamic lubrication. Let's assume that sliding occurs in the x direction only so the sliding velocity is given by $\boldsymbol{V}_0 = V_0 \boldsymbol{e}_x$. The equation then can be expanded as:

$$\left(\frac{\partial}{\partial x}\left(h^3 \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial P}{\partial y}\right)\right) = 6\mu V_0 \frac{\partial h}{\partial x}.$$

Expanding the derivatives gives:

$$3h^2 \frac{\partial h}{\partial x}\frac{\partial P}{\partial x} + h^3 \frac{\partial^2 P}{\partial x^2} + 3h^2 \frac{\partial h}{\partial y}\frac{\partial P}{\partial y} + h^3 \frac{\partial^2 P}{\partial y^2} = 6\mu V_{0x} \frac{\partial h}{\partial x}$$

At the border of the domain assuming to be in contact with the lubricant we can impose zero pressure

$$P(\Gamma) = 0,$$

as illustrated in Fig. 1: $P(x_0) = P(x_1) = 0$.

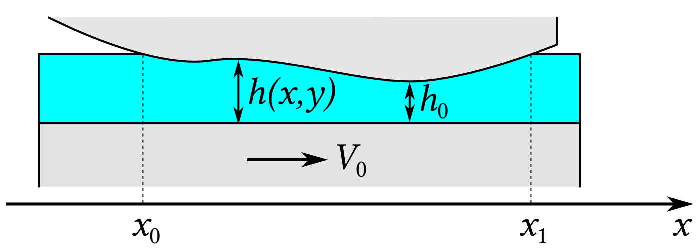

Figure 1: Schematic of the problem.

2. Finite Difference Solver

Reynolds equation can be solved on a rectangular domain $[0, L_x] \times [0, L_y]$ discretized in $N_x \times N_y$ elements. Finite difference scheme can be used to approximate all derivatives as:

$$\mathcal{H}_x + \mathcal{H}_y = 6\mu V_{0x} \frac{\partial h_{i,j} - h_{i-1,j}}{\partial x}$$

with

$$\mathcal{H}_x = 3h_{i,j}^2 \frac{\partial h_{i,j}}{\partial x} \frac{P_{i+1,j} - P_{i-1,j}}{2\Delta x} + h_{i,j}^3 \frac{P_{i+1,j} - 2P_{i,j} + P_{i-1,j}}{\Delta x^2}$$

$$\mathcal{H}_y = 3h_{i,j}^2 \frac{\partial h_{i,j}}{\partial y} \frac{P_{i,j+1} - P_{i,j-1}}{2\Delta y} + h_{i,j}^3 \frac{P_{i,j+1} - 2P_{i,j} + P_{i,j-1}}{\Delta y^2}.$$

Note that since our profile $h(x, y)$ is analytical function, we do not use finite differences to find the derivatives of h. Note also that the first order derivative of pressure could be also approximated as $\partial P/\partial x = (P_{i+1,j} - P_{i,j})/\Delta x$.

The linear system of equations can thus be formed

$$\left[M\right] [P] = [B],$$

where vector $[P]$ is a flattened vector of pressures on a regular grid

$$[P] = \left[P_{11}, P_{12}, \ldots, P_{1n_y}, P_{21}, P_{22}, \ldots, P_{2n_y}, \ldots, P_{n_x 1}, P_{n_x 2}, \ldots, P_{n_x n_y}\right]^\intercal,$$

$[B]$ is the right hand side and $[M]$ is the five-diagonal matrix of coefficients. The components of the $[M]$ matrix are the following split into five diagonals:

- Central terms in front of $P_{i,j}$:

$$A_{ij} = -2h_{i,j}^3 \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)$$

- “East” terms in front of $P_{i+1,j}$:

$$B_{i+1,j} = \frac{3h_{i,j}^2}{2\Delta x} \frac{\partial h_{i,j}}{\partial x} + \frac{h_{i,j}^3}{\Delta x^2}$$

- “North” terms in front of $P_{i,j+1}$:

$$C_{i,j+1} = \frac{3h_{i,j}^2}{2\Delta y} \frac{\partial h_{i,j}}{\partial y} + \frac{h_{i,j}^3}{\Delta y^2}$$

- “West” terms in front of $P_{i-1,j}$:

$$B_{i-1,j} = -\frac{3h_{i,j}^2}{2\Delta x} \frac{\partial h_{i,j}}{\partial x} + \frac{h_{i,j}^3}{\Delta x^2}$$

- “South” terms in front of $P_{i,j-1}$:

$$C_{i,j-1} = -\frac{3h_{i,j}^2}{2\Delta y} \frac{\partial h_{i,j}}{\partial y} + \frac{h_{i,j}^3}{\Delta y^2}$$

So the structure of the matrix is given by:

$$[M] = \begin{bmatrix} A_{11} & C_{12} & 0 & \cdots & 0 & B_{21} & 0 & 0 & \cdots & 0 \\ C_{11} & A_{12} & C_{13} & \cdots & 0 & 0 & B_{22} & 0 & \cdots & 0 \\ 0 & C_{12} & A_{13} & \cdots & 0 & 0 & 0 & B_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{1n_y} & 0 & 0 & 0 & \cdots & 0 \\ B_{11} & 0 & 0 & \cdots & 0 & A_{21} & C_{22} & 0 & \cdots & 0 \\ 0 & B_{12} & 0 & \cdots & 0 & A_{22} & C_{23} & 0 & \cdots & 0 \\ 0 & 0 & B_{13} & \cdots & 0 & C_{22} & A_{23} & C_{24} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A_{n_x n_y} \end{bmatrix}$$

As can be seen, the matrix has only five diagonals with non-zero elements, therefore a sparse matrix should be used to store it.

Boundary conditions

Now, to impose zero Dirichlet boundary conditions on the border of the domain, we can use penalty method: i.e. set big values at the diagonal of the $[M]$ matrix. If $I = \{i_1 + n_x j_1, i_2 + n_x j_2, \ldots, i_k + n_x j_k\}$ is the set of indices of the nodes on the border of the domain, then the diagonal elements of the $[M]$ matrix should be set to a large number α. The right hand side $[B]$ should be set to zero at these nodes. To impose non-zero pressure at the inlet, we can set the corresponding element of the right-hand side vector $[B]$ to αP_{inlet}, where P_{inlet} is the inlet pressure.

Flux

As soon as the pressure field is found, one can compute the flux as

$$\boldsymbol{q} = -\frac{h^3}{12\mu}\nabla P$$

or in component form

$$q_x = -\frac{h^3}{12\mu}\frac{\partial P}{\partial x}, \quad q_y = -\frac{h^3}{12\mu}\frac{\partial P}{\partial y}.$$

3. Practical work

You are provided with two python solvers:

- `1DReynoldsSolver.py` - 1D Reynolds equation solver
- `ReynoldsSolverSparseOptimized.py` - 2D Reynolds equation solver

To run them type

```
python3 1DReynoldsSolver.py
```

or simply

```
python 1DReynoldsSolver.py
```

3.1 Fixed-Incline Slider Bearing

Consider a fixed-incline slider bearing shown in Fig.2. In file `1DReynoldsSolver.py`, you will find a finite difference solver to solve Reynolds equation for this configuration. It solves equation:

$$3h^2\frac{\partial h}{\partial x}\frac{\partial P}{\partial x} + h^3\frac{\partial^2 P}{\partial x^2} = 6\mu V_{0x}\frac{\partial h}{\partial x}$$

for $x \in [0,1]$ and boundary conditions $P(0) = P(1) = 0$.

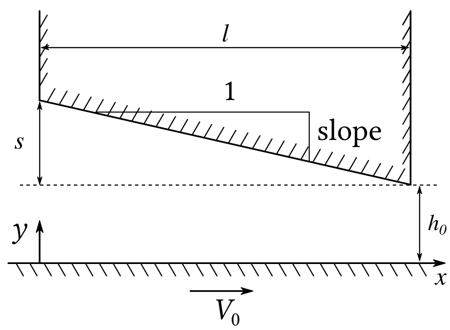

Figure 2: Fixed-Incline Slider Bearing

3.1.1 Task

1. Solve analytically Reynolds equation for fixed-incline slider bearing. Implement your analytical solution in function:

```
def analyticalSolution(x,h0,l,slope,mu,v):
  return your_solution
```

2. Compare your solution with the numerical solution.
3. Adjust parameters of the numerical solution to increase the accuracy.
4. Study the change of solution as a function of slider's slope and minimal film thickness h_0.

3.2 3D Fixed-Incline Cylindrical Slider Bearing

Consider a fixed-inline cylindrical slider bearing with

$$z(x,y) = \begin{cases} h0 - s(x - x_0 - R), & \text{if } (x-x_0)^2 + (y-y_0)^2 \le R^2, \\ h_{\max}, & \text{elsewhere.} \end{cases}$$

We solve Reynolds equation:

$$3h^2\frac{\partial h}{\partial x}\frac{\partial P}{\partial x} + h^3\frac{\partial^2 P}{\partial x^2} + 3h^2\frac{\partial h}{\partial y}\frac{\partial P}{\partial y} + h^3\frac{\partial^2 P}{\partial y^2} = 6\mu V_{0x}\frac{\partial h}{\partial x}$$

At the border of the domain assuming to be in contact with the lubricant we can impose zero pressure

$$P((x-x_0)^2 + (y-y_0)^2 \geq R^2) = 0.$$

Python script `ReynoldsSolverSparseOptimized.py` solves this problem using finite differences as shown in Section 1. It plots pressure field $P(x, y)$ and streamlines of the flux $\boldsymbol{q} = q_x\boldsymbol{e}_x + q_y\boldsymbol{e}_y$.

3.2.1 Task

1. Play with physical and numerical parameters in the solver to understand better the physics of such bearings.
2. Compare qualitatively the solution of the 3D problem with the 2D one.
3. Compare quantitative solutions for 3D and 2D cases for the pressure along the central line.
4. Add surface roughness to the incline sliding bearing. How does it change pressure distribution?