Practical Work: Viscoelastic indentation

Vladislav A. Yastrebov

January 2024

1. Introduction

Boussinesq solution for the relationship between normal force N applied at x', y' and the normal displacement of an elastic isotropic half-space is given by the following equation:

$$u_z(x,y) = \frac{(1-\nu)N}{2\pi G r}, \quad r = \sqrt{(x-x')^2 + (y-y')^2}$$

where G is the shear modulus and ν is the Poisson ratio. This solution could be generalized for the case of viscoelastic half-space through Laplace transform. The Laplace transform of the Boussinesq solution taking into account the loading history is given by:

$$\bar{u}_z(x,y,) = \frac{(1-\nu)s\bar{J}(s)\bar{N}(s)}{2\pi r},$$

where the Laplace transform of the loading history is given by:

$$\bar{N}(s) = \int_0^t N(t')e^{-s(t-t')}dt',$$

and $J(t)$ is the creep compliance function. The creep compliance function is given by:

$$J(t) = \left[\frac{1}{E_\infty} + \left(\frac{1}{E_0} - \frac{1}{E_\infty}\right)\left(1 - e^{-\frac{t}{\tau}}\right)\right] H(t),$$

where $H(t)$ is the Heaviside step function, E_0 is the instantaneous Young's modulus and E_∞ is the long-term Young's modulus, and τ is the characteristic relaxation time. The Laplace transform of the creep compliance function is given by:

$$\bar{J}(s) = \frac{1}{s}\left[\frac{1}{E_\infty} + \frac{1}{E_0 - E_\infty}\frac{\tau s}{1+\tau s}\right].$$

Integrating the evolution of the pressure distribution in space in time and using the convolution theorem, we obtain the following expression for the normal displacement:

$$u_z(x,y,t) = \frac{(1-\nu)}{2\pi}\int_{\Omega_m} \frac{q(x,y,a)}{r} dA, \tag{1}$$

where

$$q(x, y, a) = J * dP \tag{2}$$

is the convolution of the creep compliance function and the pressure distribution in space.

2. Viscoeastic indentation

In this practical work, we will consider the case of a rigid spherical indenter of radius R and a viscoelastic half-space. Let us assume that the indenter is pushed into the half-space with a monotonic displacement prescribed in time $\delta(t)$. Then the contact radius $a(t)$ will change in time, but within the contact area, displacement distribution is known:

$$\text{for } r \le a(t): \quad u_z(x, y, t) = \delta(t) - \frac{x^2 + y^2}{2R} H(t).$$

Substituting this expression into Eq. (1), we obtain:

$$\delta(t) - \frac{x^2 + y^2}{2R} = \frac{(1 - \nu)}{2\pi} \int_{\Omega(t)} \frac{q(x, y, a)}{r} dA.$$

We replaced maximal contact area Ω_m by $\Omega(t)$ assuming that the contact area is monotonically increasing function. Since contact radius $a(t) = \sqrt{\delta(t)R}$, we can show that the above equation results in the following form of q:

$$q = \begin{cases} \dfrac{4}{\pi(1 - \nu)R} \sqrt{\delta(t)R - r^2}, & \text{if } r \le a(t) \\ 0, & \text{elsewhere} \end{cases} = \frac{4}{\pi(1 - \nu)R} \mathrm{Re}\{\sqrt{\delta(t)R - r^2}\},$$

where Re designates real part. This form together with the definition of q (2) can be inverted to obtain the pressure:

$$P(x, y, t) = \frac{4}{\pi(1 - \nu)R} \int_0^t \mu(t - \tau)\, d\left[\mathrm{Re}\left\{\sqrt{\delta(\tau)R - r^2}\right\}\right],$$

where $\mu(t)$ is the deviatoric relaxation function, i.e. the relaxation function makes a link between deviatoric strain rate $\dot{e}_{ij}$ and deviatoric stress s_{ij}:

$$s_{ij} = \int_0^t \mu(t - \tau) \dot{e}_{ij}(\tau) d\tau.$$

Because of the symmetry of revolution, we can rewrite the equation for pressure as:

$$P(r, t) = \frac{4}{\pi(1 - \nu)R} \int_0^t \mu(t - \tau)\, d\left[\mathrm{Re}\left\{\sqrt{\delta(\tau)R - r^2}\right\}\right].$$

The expression under the differential sign can we rewritten as:

$$d\left[\mathrm{Re}\left\{\sqrt{\delta(\tau)R - r^2}\right\}\right] = \begin{cases} \dfrac{1}{2\sqrt{R\delta(\tau) - r^2}} \dfrac{\partial \delta(\tau)}{\partial \tau} d\tau, & \text{if } r \le a(\tau) = \sqrt{R\delta(\tau)} \\ 0, & \text{elsewhere.} \end{cases}$$

Therefore, the pressure can be written as:

$$P(r,t) = \frac{2}{\pi(1-\nu)} \int\limits_0^t \mu(t-t') \frac{\partial \delta(t')}{\partial t'} \frac{1}{\sqrt{R\delta(t') - r^2}} dt'.$$

Assuming a single relaxation time τ, we can write the deviatoric relaxation function as:

$$\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty) \exp(-t/\tau).$$

3. Numerical implementation

You are provided with Python code `Solver_ViscoelasticIndenter.py` that implements the above equations. Your task is to study indentation at different loading rates and to compare the results with the elastic case. You can control the loading rate by changing the parameter `v` in the code. For material parameters, you can control `mu0`, `mu1` for shear moduli for slow and fast loading rates, respectively, and `tau` for the relaxation time. Finally, you can consider indentation regime different from simple linear one implemented in the code: `delta = v*t`.

References

In this practical work we closely followed the book by R. Christensen: Theory of Viscoelasticity: An Introduction, Academic Press, 1982. p.160-165.